MATHEMATICAL ANALYSIS

Division of Mathematics Faculty of Education

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Chapter 1

The Real Number System

1.1 Ordered field axioms

FIELD AXIOMS.

There are functions $+$ and $\cdot$, defined on $\mathbb{R}^2$, that satisfy the following properties for every $a, b, c \in \mathbb{R}$:

F1	**Closure Properties**	$a + b$ and $a \cdot b$ belong to $\mathbb{R}$.
F2	**Associative Properties**	$a + (b + c) = (a + b) + c$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
F3	**Commutative Properties**	$a + b = b + a$ and $a \cdot b = b \cdot a$
F4	**Distributive Properties**	$a \cdot (b + c) = a \cdot b + a \cdot c$ $(b + c) \cdot a = b \cdot a + c \cdot a$
F5	**Additive Identity**	There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a = a + 0$ for all $a \in \mathbb{R}$.
F6	**Multiplicative Identity**	There is a unique element $1 \in \mathbb{R}$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in \mathbb{R}$.
F7	**Additive Inverse**	For every $x \in \mathbb{R}$ there is a unique $-x \in \mathbb{R}$ such that $x + (-x) = 0 = (-x) + x$.
F8	**Multiplicative Inverse**	For every $x \in \mathbb{R}\backslash\{0\}$ there is a unique $x^{-1} \in \mathbb{R}$ such that $x \cdot (x^{-1}) = 1 = (x^{-1}) \cdot x$.

We shall frequently denote

$$a + (-b) \text{ by } a - b, \quad a \cdot b \text{ by } ab, \quad a^{-1} \text{ by } \frac{1}{a} \quad \text{and} \quad a \cdot b^{-1} \text{ by } \frac{a}{b}.$$

The real number system $\mathbb{R}$ contains certain special subsets: the set of **natural numbers**

$$\mathbb{N} := \{1, 2, 3, ...\}$$

obtained by begining with 1 and successively adding 1's to form $2 := 1 + 1$, $3 := 2 + 1$, etc,; the set of **integers**

$$\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$$

(Zahlen is German for number); the set of **rationals** (or fractions or quoteints)

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$$

and the set of **irrationals**

$$\mathbb{Q}^c := \mathbb{R} \backslash \mathbb{Q}.$$

Equality in $\mathbb{Q}$ is defined by

$$\frac{m}{n} = \frac{p}{q} \quad \text{if and only if} \quad mq = np.$$

Recall that each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ is a proper subset of the next; i.e.,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Definition 1.1.1 *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Define*

$$a^n = \underbrace{a \cdot a \cdot ... \cdot a}_{n- \text{ copies}}$$

*a and n are called **base** and **exponent**, respectively.*

Definition 1.1.2 *Let a be a non-zero real number. Define*

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n} \quad \text{for } n \in \mathbb{N}$$

Theorem 1.1.3 *Let $a, b \in \mathbb{R}$ and $n, m \in \mathbb{Z}$. Then*

1. $(ab)^n = a^n b^n$
2. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ *where* $b \neq 0$
3. $a^n \cdot a^m = a^{m+n}$
4. $\frac{a^n}{a^m} = a^{n-m}$ *where* $a \neq 0$

Proof. Excercise. □

Theorem 1.1.4 *Let a be a real number. Then*

1. $0a = 0$
2. $(-1)a = -a$
3. $-(-a) = a$
4. $(a^{-1})^{-1} = a$ *where* $a \neq 0$

Theorem 1.1.5 *Let a and b be real numbers. Then*

$$-(ab) = a(-b) = (-a)b.$$

Theorem 1.1.6 (Cancellation) *Let a, b and c be real numbers. Then*

1. *Cancellation for addition* *if* $a + c = b + c$, *then* $a = b$.

2. *Cancellation for multiplication* *if* $ac = bc$ *and* $c \neq 0$, *then* $a = b$.

Theorem 1.1.7 (Integral Domain) *Let a and b be real numbers.*

$$\textit{If} \quad ab = 0\ , \ \textit{then} \quad a = 0 \quad \textit{or} \quad b = 0.$$

ORDER AXIOMS.

There is a relation $<$ on $\mathbb{R}^2$ that has the following properties for every $a, b, c \in \mathbb{R}$.

O1	**Trichotomy Property**	Given $a, b \in \mathbb{R}$, one and only one of the following statements holds: $a < b, \quad b < a, \quad \text{or} \quad a = b$
O2	**Trasitive Property**	$a < b$ and $b < c$ imply $a < c$
O3	**Additive Property**	$a < b$ imply $a + c < b + c$
O4	**Multiplicative Property**	O4.1 $a < b$ and $0 < c$ imply $ac < bc$ O4.2 $a < b$ and $c < 0$ imply $bc < ac$

We define in other cases:

- By $b > a$ we shall mean $a < b$.
- By $a \leq b$ we shall mean $a < b$ or $a = b$.
- If $a < b$ and $b < c$, we shall write $a < b < c$.
- We shall call a number $a \in \mathbb{R}$ **nonnegative** if $a \geq 0$ and **positive** if $a > 0$.

Example 1.1.8 *Let* $x \in \mathbb{R}$*. Show that if* $0 < x < 1$*, then* $0 < x^2 < x$

Example 1.1.9 *Let* $x, y \in \mathbb{R}$*. Show that if* $0 < x < y$*, then* $0 < x^2 < y^2$

Theorem 1.1.10 *Let* a, b, c *and* d *be real numbers.*

If $a < b$ *and* $c < d$*, then* $a + c < b + d$*.*

Theorem 1.1.11 *Let* a, b, c *and* d *be real numbers.*

If $0 < a < b$ *and* $0 < c < d$*, then* $ac < bd$*.*

Theorem 1.1.12 *If $a \in \mathbb{R}$, prove that*

$$a \neq 0 \quad \textit{implies} \quad a^2 > 0.$$

In particular, $-1 < 0 < 1$.

Example 1.1.13 *If $x \in \mathbb{R}$, prove that $x > 0$ implies $x^{-1} > 0$.*

Example 1.1.14 *If $x \in \mathbb{R}$, prove that $x < 0$ implies $x^{-1} < 0$.*

Theorem 1.1.15 *Let a and b be real numbers such that $0 < a < b$. Then*

$$\frac{1}{b} < \frac{1}{a}.$$

Example 1.1.16 *Let a and b be real numbers such that $b < a < 0$. Then*

$$\frac{1}{a} < \frac{1}{b}.$$

Example 1.1.17 *Let x and y be two distinct real numbers. Prove that*

$$\frac{x+y}{2} \text{ lies between } x \text{ and } y.$$

ABSOLUTE VALUE.

Definition 1.1.18 (**Absolute Value**) *The* ***absolute value*** *of a number $a \in \mathbb{R}$ is a the number*

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 1.1.19 (**Positive Definite**) *For all $a \in \mathbb{R}$,*

1. $|a| \geq 0$
2. $|a| = 0$ *if and only if* $a = 0$

Theorem 1.1.20 (**Multiplicative Law**) *For all $a, b \in \mathbb{R}$,*

$$|ab| = |a||b|.$$

Theorem 1.1.21 (Symmetric Law) *For all $a, b \in \mathbb{R}$,*

$$|a - b| = |b - a|.$$

Moreover, $|a| = |-a|$.

Example 1.1.22 *Show that* $\left|\frac{1}{x}\right| = \frac{1}{|x|}$ *for all $x \neq 0$.*

Theorem 1.1.23 *Let $a, b \in \mathbb{R}$. Show that*

1. $|a^2| = a^2$
2. $a \leq |a|$
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ *when* $b \neq 0$

Theorem 1.1.24 *Let $a \in \mathbb{R}$ and $M \geq 0$. Then*

$$|a| \leq M \quad \textit{if and only if} \quad -M \leq a \leq M$$

Corollary 1.1.25 *For all $a \in \mathbb{R}$,* $-|a| \leq a \leq |a|$.

INTERVAL.

Let a and b real numbers. A **closed interval** is a set of the form

$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ $\qquad$ $(-\infty,b] := \{x \in \mathbb{R} : x \leq b\}$

$[a,\infty) := \{x \in \mathbb{R} : a \leq x\}$ $\qquad$ $(-\infty,\infty) := \mathbb{R},$

and an **open interval** is a set of the form

$(a,b) := \{x \in \mathbb{R} : a < x < b\}$ $\qquad$ $(-\infty,b) := \{x \in \mathbb{R} : x < b\}$

$(a,\infty) := \{x \in \mathbb{R} : a < x\}$ $\qquad$ $(-\infty,\infty) := \mathbb{R}.$

By an **interval** we mean a closed interval, an open interval, or a set of the form

$$[a,b) := \{x \in \mathbb{R} : a \leq x < b\} \quad \text{or} \quad (a,b] := \{x \in \mathbb{R} : a < x \leq b\}$$

Notice, then, that when $a < b$, then intervals $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) correspond to line segments on the real line, but when $b < a$, these interval are all the empty set.

Example 1.1.26 *Solve* $|x-1| \leq 1$ *for* $x \in \mathbb{R}$ *in interval form.*

Example 1.1.27 *Show that if* $|x| < 1$*, then* $|x^2 + x| < 2$*.*

Example 1.1.28 *Show that if* $|x-1|<2$, *then* $\dfrac{1}{|x|}>1$.

Theorem 1.1.29 (**Triangle Inequality**) *Let* $a, b \in \mathbb{R}$. *Then,*

$$|a+b| \leq |a| + |b|.$$

Theorem 1.1.30 (**Apply Triangle Inequality**) *Let $a, b \in \mathbb{R}$. Then,*

1. $|a - b| \leq |a| + |b|$
2. $|a| - |b| \leq |a - b|$
3. $|a| - |b| \leq |a + b|$
4. $||a| - |b|| \leq |a - b|$

Example 1.1.31 *Show that if* $|x - 2| < 1$, *then* $|x| < 3$.

Theorem 1.1.32 *Let $x, y \in \mathbb{R}$. Then*

1. $x < y + \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \leq y$

2. $x > y - \varepsilon$ *for all* $\varepsilon > 0$ *if and only if* $x \geq y$

Corollary 1.1.33 *Let $a \in \mathbb{R}$. Then*

$$|a| < \varepsilon \quad \text{for all } \varepsilon > 0 \quad \text{if and only if} \quad a = 0$$

Exercises 1.1

1. Let $a, b \in \mathbb{R}$. Prove that

   1.1 $-(a-b) = b-a$

   1.2 $a(b-c) = ab-ac$

   1.3 $(-a)(-b) = ab$

   1.4 $\dfrac{-a}{b} = \dfrac{a}{-b} = -\dfrac{a}{b}$ when $b \neq 0$

2. Let $a, b \in \mathbb{R}$. Prove that

   2.1 If $a+b=a$, then $x=0$.

   2.2 If $ab=b$ and $b \neq 0$, then $a=1$.

   2.3 If $a^{-1}=a$ and $a \neq 0$, then $a=-1$ or $a=1$.

3. Let $a, b, c, d \in \mathbb{R}$. Prove that

   3.1 if $a<b<0$, then $0<b^2<a^2$.

   3.2 if $a \leq b$ and $a \geq b$, then $a=b$.

   3.3 if $0<a<b$, then $\sqrt{a}<\sqrt{b}$.

4. Solve each of the following inequality for $x \in \mathbb{R}$.

   4.1 $|1-2x| \leq 3$

   4.2 $|3-x| < 5$

   4.3 $|x^2-x-1| < x^2$

   4.4 $|x^2-x| < 2$

5. Prove that if $0<a<1$ and $b=1-\sqrt{1-a}$, then $0<b<a$.

6. Prove that if $a>2$ and $b=1-\sqrt{1-a}$, then $2<b<a$.

7. Prove that $|x| \leq 1$ implies $|x^2-1| \leq 2|x-1|$.

8. Prove that $-1 \leq x \leq 2$ implies $|x^2+x-2| \leq 4|x-1|$.

9. Prove that $|x| \leq 1$ implies $|x^2-x-2| \leq 3|x+1|$.

10. Prove that $0<|x-1| \leq 1$ implies $|x^3+x-2| < 8|x-1|$. Is this true if $0 \leq |x-1| < 1$?

11. Let $x, y \in \mathbb{R}$. Prove that if $|x+y| = |x-y|$, then $x|y| + y|x| = 0$.

12. Let $x, y \in \mathbb{R}$. Prove that if $|2x+y| = |x+2y|$, then $|xy| = x^2$.

13. Let $a \in \mathbb{R}$. Prove that $\dfrac{a^2+2}{\sqrt{a^2+1}} \geq 2$.

14. Prove that
$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$$
for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$

15. Let $x, y \in \mathbb{R}$. Prove that $x > y - \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.

16. Suppose that $x, a, y, b \in \mathbb{R}$, $|x-a| < \varepsilon$ and $|y-b| < \varepsilon$ for some $\varepsilon > 0$. Prove that

    16.1 $|xy - ab| < (|a| + |b|)\varepsilon + \varepsilon^2$

    16.2 $|x^2y - a^2b| < \varepsilon(|a|^2 + 2|ab|) + \varepsilon^2(|b| + 2|a|) + \varepsilon^2$

17. The **positive part** of an $a \in \mathbb{R}$ is defined by
$$a^+ := \frac{|a| + a}{2}$$
and the **negative part** by
$$a^- := \frac{|a| - a}{2}.$$

    17.1 Prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$.

    17.2 Prove that $a^+ := \begin{cases} a & : a \geq 0 \\ 0 & : a \leq 0 \end{cases}$ and $a^- := \begin{cases} 0 & : a \geq 0 \\ -a & : a \leq 0 \end{cases}$.

18. Let $a, b \in \mathbb{R}$. The **arithmetic mean** of a, b is $A(a,b) := \dfrac{a+b}{2}$,
the **geometric mean** of $a, b \in (0, \infty)$ is $G(a,b) := \sqrt{ab}$,
and **harmonic mean** of $a, b \in (0, \infty)$ is $H(a,b) := \dfrac{2}{a^{-1} + b^{-1}}$.
Show that

    18.1 if $a, b \in (0, \infty)$. Then $H(a,b) \leq G(a,b) \leq A(a,b)$.

    18.2 if $0 < a \leq b$. Then $a \leq G(a,b) \leq A(a,b) \leq b$.

    18.3 if $0 < a \leq b$. Then, $G(a,b) = A(a,b)$ if and only if $a = b$.

1.2 Well-Ordering Principle

Definition 1.2.1 *A number m is a **least element** of a set $S \subset \mathbb{R}$ if and only if*

$$m \in S \text{ and } m \leq s \text{ for all } s \in S.$$

WELL-ORDERING PRINCIPLE (WOP).

Every nonempty subset of $\mathbb{N}$ has a least element.

$$S \subseteq \mathbb{N} \wedge S \neq \varnothing \ \rightarrow \ \exists m \in S \, \forall s \in S, \ m \leq s.$$

Theorem 1.2.2 (**Mathematical Induction**) *Suppose for each $n \in \mathbb{N}$ that $P(n)$ is a statement that satisfies the following two properties:*

(1) Basic step : $P(1)$ is true

(2) Inductive step : For every $k \in \mathbb{N}$ for which $P(k)$ is true, $P(k+1)$ is also true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1.2.3 (**Gauss' formula**) *Prove that*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Example 1.2.4 *Prove that* $2^n > n$ *for all* $n \in \mathbb{N}$.

BINOMIAL FORMULA.

Definition 1.2.5 *The notation* $0! = 1$ *and* $n! = 1 \cdot 2 \cdots (n-1) \cdot n$ *for* $n \in \mathbb{N}$ *(called* ***factorial****), define the* ***binomial coefficient n over k*** *by*

$$\binom{n}{k} := \frac{n!}{(n-k)!k!}$$

for $0 \leq k \leq n$ and $n = 0, 1, 2, 3, ...$

Theorem 1.2.6 *If* $n, k \in \mathbb{N}$ *and* $1 \leq k \leq n$, *then*

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Theorem 1.2.7 (Binomial formula) *If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Exercises 1.2

1. Prove that the following formulas hold for all $n \in \mathbb{N}$.

   1.1 $\displaystyle\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n$

   1.2 $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

   1.3 $\displaystyle\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$

   1.4 $\displaystyle\sum_{k=1}^{n} \frac{a-1}{a^k} = 1 - \frac{1}{a^n}, \quad a \neq 0$

2. Use the Binomial Formula to prove each of the following.

   2.1 $\displaystyle 2^n = \sum_{k=1}^{n} \binom{n}{k}$ for all $n \in \mathbb{N}$.

   2.2 $(a+b)^n \geq a^n + aa^{n-1}b$ for all $n \in \mathbb{N}$ and $a, b \geq 0$.

   2.3 $\displaystyle\left(1 + \frac{1}{n}\right)^n \geq 2$ for all $n \in \mathbb{N}$.

3. Let $n \in \mathbb{N}$. Write

$$\frac{(x+h)^n - x^n}{h}$$

   as a sum none of whose terms has an h in the dennominator.

4. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_{n+1} < x_n < 1$ holds for all $n \in \mathbb{N}$.

5. Suppose that $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $2 \leq x_{n+1} \leq x_n \leq x_1$ holds for all $n \in \mathbb{N}$.

6. Suppose that $0 < x_1 < 2$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Prove that $0 < x_n < x_{n+1} < 2$ holds for all $n \in \mathbb{N}$.

7. Prove that each of the following inequalities hold for all $n \in \mathbb{N}$.

   7.1 $n < 3^n$ 7.2 $n^2 \leq 2^n + 1$ 7.3 $n^3 \leq 3^n$

8. Let $0 < |a| < 1$. Prove that $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$.

9. Prove that $0 \leq a < b$ implies $a^n < b^n$ for all $n \in \mathbb{N}$.

1.3 Completeness Axiom

SUPREMUM.

Definition 1.3.1 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be* ***bounded above*** *if and only if*

$$\text{there is an } M \in \mathbb{R} \text{ such that } a \leq M \text{ for all } a \in A$$

2. *A number M is called an* ***upper bound*** *of the set A if and only if*

$$a \leq M \text{ for all } a \in A$$

3. *A number s is called a* ***supremum*** *of the set A if and only if*

$$s \text{ is an upper bound of } A \text{ and } s \leq M \text{ for all upper bound } M \text{ of } A$$

In this case we shall say that A has a supremum s and shall write $s = \sup A$

Example 1.3.2 *Fill the blanks of the following table.*

Sets	Bounded above	Set of Upper bound	Supremum
$A = [0, 1]$			
$A = (0, 1)$			
$A = \{1\}$			
$A = (0, \infty)$			
$A = (-\infty, 0)$			
$A = \mathbb{N}$			
$A = \mathbb{Z}$			

Example 1.3.3 *Show that* $\sup A = 1$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Theorem 1.3.4 *If a set has one upper bound, then it has infinitely many upper bounds.*

Theorem 1.3.5 *If a set has a supremum, then it has only one supremum.*

Theorem 1.3.6 (**Approximation Property for Supremum (APS)**) *If A has a supremum and $\varepsilon > 0$ is any positive number, then there is a point $a \in A$ such that*

$$\sup A - \varepsilon < a \leq \sup A$$

Theorem 1.3.7 *If $A \subset \mathbb{N}$ has a supremum, then* $\sup A \in A$.

COMPLETENESS AXIOM.

If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then A has a supremum.

Theorem 1.3.8 *The set of natural numbers is not bounded above.*

Theorem 1.3.9 (**Archimedean Properties (AP)**) *For each $x \in \mathbb{R}$, the following statements are true.*

1. *There is an integer $n \in \mathbb{N}$ such that $x < n$.*
2. *If $x > 0$, there there is an integer $n \in \mathbb{N}$ such that $\dfrac{1}{n} < x$.*

Theorem 1.3.10 *Let $x \in \mathbb{R}$. Then*

$$|x| < \frac{1}{n} \quad \textit{for all } n \in \mathbb{N} \quad \textit{if and only if} \quad x = 0$$

Example 1.3.11 *Let* $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

Example 1.3.12 *Let* $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\sup A = 1$.

Theorem 1.3.13 *If* $x \in \mathbb{R}$*, then there is an* $n \in \mathbb{Z}$ *such that*

$$n - 1 \leq x < n.$$

Theorem 1.3.14 (**Density of Rationals**) *If* $a, b \in \mathbb{R}$ *satisfy* $a < b$*, then there is a rational number* r *such that*

$$a < r < b.$$

Theorem 1.3.15 $\sqrt{2}$ *is irrational.*

Theorem 1.3.16 (**Density of Irratioals**) *If* $a, b \in \mathbb{R}$ *satisfy* $a < b$*, then there is an irrational number* t *such that*

$$a < t < b.$$

INFIMUM.

Definition 1.3.17 *Let A be a nonempty subset of $\mathbb{R}$.*

1. *The set A is said to be **bounded below** if and only if*

$$\text{there is an } m \in \mathbb{R} \text{ such that } m \leq a \text{ for all } a \in A$$

2. *A number m is called a **lower bound** of the set A if and only if*

$$m \leq a \quad \text{for all } a \in A$$

3. *A number ℓ is called an **infimum** of the set A if and only if*

$$\ell \text{ is a lower bound of } A \text{ and } m \leq \ell \text{ for all lower bound } m \text{ of } A$$

   In this case we shall say that A has an infimum s and shall write $\ell = \inf A$

4. *A is said to be **bounded** if and only if it is bounded above and below.*

Example 1.3.18 *Fill the blanks of the following table.*

Sets	Bounded below	Set of Lower bound	Infimum	Bounded
$A = [0, 1]$				
$A = (0, 1)$				
$A = \{1\}$				
$A = (0, \infty)$				
$A = (-\infty, 0)$				
$A = \mathbb{N}$				
$A = \mathbb{Z}$				

Example 1.3.19 *Show that* $\inf A = 0$ *where*

1. $A = [0, 1]$
2. $A = (0, 1)$

Example 1.3.20 *Let* $A = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = 0$.

Example 1.3.21 *Let* $A = \left\{ \dfrac{n}{n+1} : n \in \mathbb{N} \right\}$. *Prove that* $\inf A = \dfrac{1}{2}$.

Theorem 1.3.22 (**Approximation Property for Infimum (API)**) *If* A *has an infimum and* $\varepsilon > 0$ *is any positive number, then there is a point* $a \in A$ *such that*

$$\inf A \leq a < \inf A + \varepsilon.$$

Exercises 1.3

1. Find the infimum and supremum of each the following sets.

1.1 $A = [0, 2)$

1.2 $A = \{4, 3, 1, 5\}$

1.3 $A = \{x \in \mathbb{R} : |x - 1| < 2\}$

1.4 $A = \{x \in \mathbb{R} : |x + 1| < 1\}$

1.5 $A = \{1 + (-1)^n : n \in \mathbb{N}\}$

1.6 $A = \left\{\frac{1}{n} - (-1)^n : n \in \mathbb{N}\right\}$

1.7 $A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

1.8 $A = \left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}$

1.9 $A = \left\{\frac{n^2+n}{n^2+1} : n \in \mathbb{N}\right\}$

1.10 $A = \left\{\frac{n(-1)^n+1}{n+2} : n \in \mathbb{N}\right\}$

2. Find $\inf A$ and $\sup A$ with proving them.

2.1 $A = [-1, 1]$

2.2 $A = (-1, 2]$

2.3 $A = (-1, 0) \cup (1, 2)$

2.4 $A = \{1, 2, 3\}$

2.5 $A = \left\{\frac{n}{n+2} : n \in \mathbb{N}\right\}$

2.6 $A = \left\{\frac{n-2}{n+2} : n \in \mathbb{N}\right\}$

2.7 $A = \left\{\frac{n}{n^2+1} : n \in \mathbb{N}\right\}$

2.8 $A = \{(-1)^n : n \in \mathbb{N}\}$

3. Let $A = \left\{1 - \frac{n}{n^2+2} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

4. Let $A = \left\{2 - \frac{n}{n^2+1} : n \in \mathbb{N}\right\}$. What are supremum and infimum of A ? Verify (proof) your answers.

5. If a set has one lower bound, then it has infinitely many lower bounds.

6. Prove that if A is a nonempty bounded subset of $\mathbb{Z}$, then both $\sup A$ and $\inf A$ exist and belong to A.

7. Prove that for each $a \in \mathbb{R}$ and each $n \in \mathbb{N}$ there exists a rational r_n such that

$$|a - r_n| < \frac{1}{n}.$$

8. Let r be a rational number and s be an irrational number. Prove that

   8.1 $r + s$ is an irrational number.

   8.2 if $r \neq 0$, then rs is always an irrational number.

9. Let $\sqrt{K} \in \mathbb{Q}^c$ and $a, b, x, y \in \mathbb{Z}$. Prove that

$$\text{if } a + b\sqrt{K} = x + y\sqrt{K}, \text{ then } a = x \text{ and } b = y.$$

10. Show that a lower bound of a set need not be unique but the infimum of a given set A is unique.

11. Show that if A is a nonoempty subset of $\mathbb{R}$ that is bounded below, then A has a finite infimum.

12. Prove that if x is an upper bound of a set $A \subseteq \mathbb{R}$ and $x \in A$, then x is the supremum of A.

13. Suppose $E, A, B \subset \mathbb{R}$ and $E = A \cup B$. Prove that if E has a supremum and both A and B are nonempty, then Sup$A$ and $\sup B$ both exist, and $\sup E$ is one of the numbers Sup$A$ or $\sup B$.

14. (**Monotone Property**) Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$. Prove that

    14.1 if B has a supremum, then $\sup A \leq \sup B$

    14.2 if B has an infimum, then $\inf B \leq \inf A$

15. Define the **reflection** of a set $A \subseteq \mathbb{R}$ by

$$-A := \{-x : x \in A\}$$

Let $A \subseteq \mathbb{R}$ be nonempty. Prove that

15.1 A has a supremum if and only if $-A$ has and infimum, in which case

$$\inf(-A) = -\sup A.$$

15.2 A has an infimum if and only if $-A$ has and supremum, in which case

$$\sup(-A) = -\inf A.$$

1.4 Functions and Inverse functions

Review notation $f : X \to Y$ that means a fuction form X to Y, each $x \in X$ is assigned a unique $y = f(x) \in Y$, there is nothing that keeps two x's from being assigned to the same y, and nothing that say every $y \in Y$ corresponds to some $x \in X$, i.e., f is a fuction if and only if for each $(x_1, y_1), (x_2, y_2)$ belong to f,

$$\text{if } x_1 = x_2 \text{ , then } y_2 = y_2.$$

Definition 1.4.1 *Let f be a function from a set X into a set Y.*

1. *f is said to be **one-to-one (1-1)** on X if and only if*

$$x_1, x_2 \in X \textit{ and } f(x_1) = f(x_2) \textit{ imply } x_1 = x_2.$$

2. *f is said to take X **onto** Y if and only if*

$$\textit{for each } y \in Y \textit{ there is an } x \in X \textit{ such that } y = f(x).$$

Example 1.4.2 *Show that $f(x) = 2x + 1$ is 1-1 from $\mathbb{R}$ onto $\mathbb{R}$.*

Theorem 1.4.3 *Let X and Y be sets and $f : X \to Y$. Then f is 1-1 from X onto Y if and only if there is a unique function g from Y onto X that satisfies*

1. $f(g(y)) = y, \quad y \in Y$

   and

2. $g(f(x)) = x, \quad x \in X$

If f is 1-1 from a set X onto a set Y, we shall say that f has an **inverse function**. We shall call the function g given in Theorem 1.4.3 the **inverse** of f, and denote it by f^{-1}. Then

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Example 1.4.4 *Find inverse function of* $f(x) = 2x + 1$.

Example 1.4.5 *Let* $f(x) = e^x - e^{-x}$.

1. *Show that* f *is 1-1 from* $\mathbb{R}$ *onto* $\mathbb{R}$.

2. *Find a formula of* $f^{-1}(x)$.

Exercises 1.4

1. For each of the following, prove f is 1-1 from A onto A. Find a formula for f^{-1}.

   1.1 $f(x) = 3x - 7$: $A = \mathbb{R}$

   1.2 $f(x) = x^2 - 2x - 1$: $A = (1, \infty)$

   1.3 $f(x) = 3x - |x| + |x - 2|$: $A = \mathbb{R}$

   1.4 $f(x) = x|x|$: $A = \mathbb{R}$

   1.5 $f(x) = e^{\frac{1}{x}}$: $A = (0, \infty)$

   1.6 $f(x) = \tan x$: $A = (-\frac{\pi}{2}, \frac{\pi}{2})$

   1.7 $f(x) = \dfrac{x}{x^2 + 1}$: $A = [-1, 1]$

2. Let $f(x) = x^2 e^{x^2}$ where $x \in \mathbb{R}$. Show that f is 1-1 on $(0, \infty)$.

3. Suppose that A is finite and f is 1-1 from A onto B. Prove that B is finite.

4. Prove that there a fuction f that is 1-1 from $\{2, 4, 6, ...\}$ onto $\mathbb{N}$.

5. Prove that there a fuction f that is 1-1 from $\{1, 3, 5, ...\}$ onto $\mathbb{N}$.

6. Suppose that $n \in \mathbb{N}$ and $\phi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$.

   6.1 Prove that ϕ is 1-1 if and only if ϕ in onto.

   6.2 Suppose that A is finite and $f : A \to A$. Prove that

   $$f \text{ is 1-1 on } A \text{ if and only if } f \text{ takes } A \text{ onto } A.$$

7. Let $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ be a 1-1 function. Show that $\sum_{x=1}^{n} f(x) = n!$.

Chapter 2

Sequences in $\mathbb{R}$

2.1 Limits of sequences

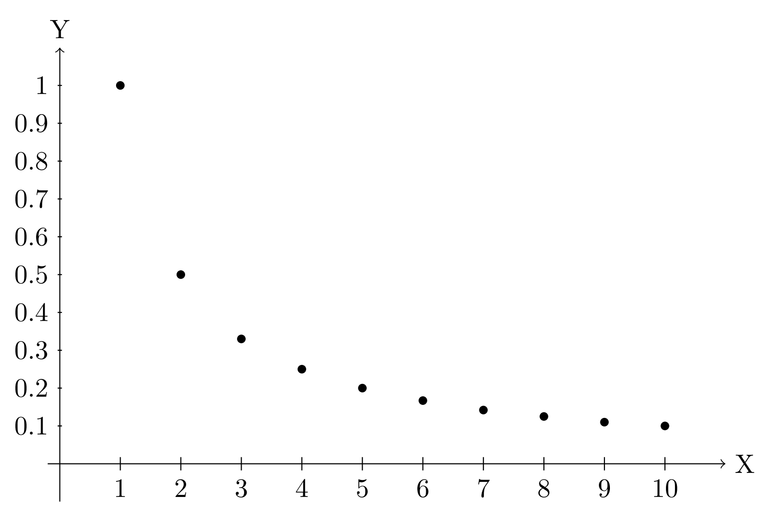
An **infinite sequence** (more briefly, a sequence) is a function whose domain in $\mathbb{N}$. A sequence f whose term are $x_n := f(n)$ will be defined by

$$x_1, x_2, x_3, \ldots \quad \text{or} \quad \{x_n\}_{n\in\mathbb{N}} \quad \text{or} \quad \{x_n\}_{n=1}^{\infty} \quad \text{or} \quad \{x_n\}.$$

Example 2.1.1 *Use notation to represents the following sequences.*

1. $1, 2, 3, \ldots$ represents the sequence $\{n\}_{n\in\mathbb{N}}$
2. $1, -1, 1, -1, \ldots$ represents the sequence $\{(-1)^n\}$

Example 2.1.2 *Sketch graph of* $\{x_n\}$ *and guess* x_n *if* n *go to infinity where* $x_n = \dfrac{1}{n}$



Definition 2.1.3 *A sequence of real numbers $\{x_n\}$ is said to* ***converge*** *to a real number $a \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad |x_n - a| < \varepsilon.$$

We shall use the following phrases and notations interchangeably:

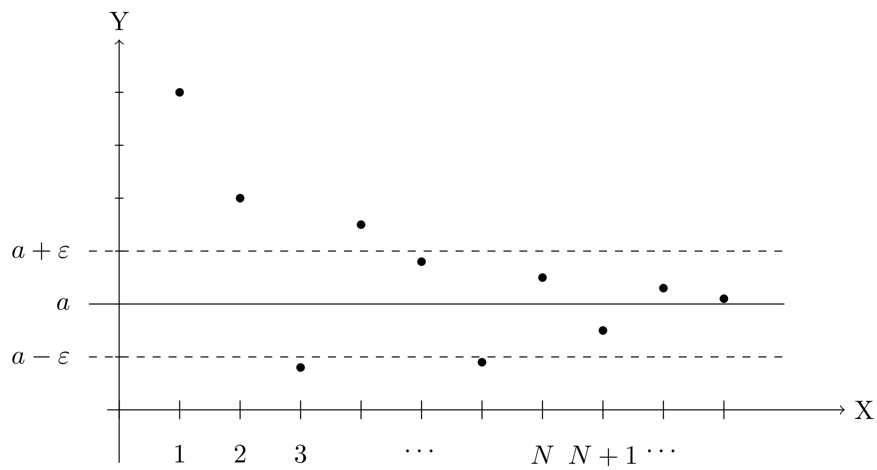
(a) $\{x_n\}$ converges to a;

(b) x_n converges to a;

(c) $\lim_{n\to\infty} x_n = a$;

(d) $x_n \to a$ as $n \to \infty$;

(e) the limit of $\{x_n\}$ exists and equals a.



Theorem 2.1.4 $\lim_{n\to\infty} k = k$ *where k is a constant.*

Example 2.1.5 *Prove that* $\dfrac{1}{n} \to 0$ *as* $n \to \infty$.

Example 2.1.6 *Prove that* $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$

Example 2.1.7 *Prove that* $\dfrac{1}{2^n} \to 0$ *as* $n \to \infty$

Example 2.1.8 *Prove that* $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$

Example 2.1.9 *Prove that* $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

Example 2.1.10 *If* $x_n \to 1$ *as* $n \to \infty$. *Prove that*

$$2x_n + 1 \to 3 \text{ as } n \to \infty.$$

Example 2.1.11 *If* $x_n \to -1$ *as* $n \to \infty$. *Prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Example 2.1.12 *Assume that* $x_n \to 1$ *as* $n \to \infty$. *Show that*

$$\frac{1}{x_n} \to 1 \text{ as } n \to \infty.$$

Example 2.1.13 *Assume that* $x_n \to 1$ *as* $n \to \infty$. *Show that*

$$\frac{1+(x_n)^2}{x_n+1} \to 1 \text{ as } n \to \infty$$

Theorem 2.1.14 *A sequence can have at most one limit.*

Example 2.1.15 *Show that the limit* $\{(-1)^n\}_{n\in\mathbb{N}}$ *has no limit or does not exist (DNE).*

SUBSEQUENCES.

Definition 2.1.16 *By a* ***subsequence*** *of a sequence* $\{x_n\}_{n\in\mathbb{N}}$*, we shall mean a sequence of the form*

$$\{x_{n_k}\}_{k\in\mathbb{N}}, \quad \textit{where each } n_k \in \mathbb{N} \textit{ and } n_1 < n_2 < n_3 < \ldots$$

Example 2.1.17 *Give examples for two subsequences of the following sequences.*

Sequences	Subsequences
$1, -1, 1, -1, 1, -1, \ldots$	
$\{n\}_{n\in\mathbb{N}}$	

Theorem 2.1.18 *If* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* a *and* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *is any subsequence of* $\{x_n\}_{n\in\mathbb{N}}$*, then*

$$x_{n_k} \textit{ converges to } a \textit{ as } k \to \infty.$$

Example 2.1.19 *Show that the limit* $\{\cos(n\pi)\}_{n\in\mathbb{N}}$ *has no limit.*

BOUNDED SEQUENCES.

Definition 2.1.20 *Let* $\{x_n\}$ *be a sequence of real numbers.*

1. $\{x_n\}$ *is said to be* ***bounded above*** *if and only if*

$$\text{there is an } M \in \mathbb{R} \text{ such that } \quad x_n \leq M \quad \text{for all } n \in \mathbb{N}$$

2. $\{x_n\}$ *is said to be* ***bounded below*** *if and only if*

$$\text{there is an } m \in \mathbb{R} \text{ such that } \quad m \leq x_n \quad \text{for all } n \in \mathbb{N}$$

3. $\{x_n\}$ *is said to be* ***bounded*** *if and only if it is both above and below or*

$$\text{there a } K > 0 \text{ such that } \quad |x_n| \leq K \quad \text{for all } n \in \mathbb{N}$$

Example 2.1.21 *Show that the following sequence is bounded above or bounded below or bounded.*

Sequences	Bounded below	Bounded above	Bounded
$\{n\}_{n\in\mathbb{N}}$			
$\{-n\}_{n\in\mathbb{N}}$			
$\{(-1)^n\}_{n\in\mathbb{N}}$			

Theorem 2.1.22 (**Bounded Convergent Theorem (BCT)**) *Every convergent sequence is bounded.*

Example 2.1.23 *Show that the limit* $\{n\}_{n\in\mathbb{N}}$ *does not exist.*

Example 2.1.24 *Assume that $x_n \to 1$ as $n \to \infty$. Use BCT to prove that*

$$(x_n)^2 \to 1 \text{ as } n \to \infty.$$

Exercises 2.1

1. Prove that the following limit exist.

1.1 $3+\dfrac{1}{n}$	as $n \to \infty$	1.5 $\dfrac{5+n}{n^2}$	as $n \to \infty$
1.2 $2\left(1-\dfrac{1}{n}\right)$	as $n \to \infty$	1.6 $\pi - \dfrac{3}{\sqrt{n}}$	as $n \to \infty$
1.3 $\dfrac{2n+1}{1-n}$	as $n \to \infty$	1.7 $\dfrac{n(n+2)}{n^2+1}$	as $n \to \infty$
1.4 $\dfrac{n^2-1}{n^2}$	as $n \to \infty$	1.8 $\dfrac{n}{n^3+1}$	as $n \to \infty$

2. Suppose that x_n is sequence of real numbers that converges to 2 as $n \to \infty$.

   Use Definition 2.1.3, prove that each of the following limit exists.

2.1 $2 - x_n \to 0$	as $n \to \infty$	2.4 $\dfrac{1}{x_n - 1} \to 1$	as $n \to \infty$
2.2 $3x_n + 1 \to 7$	as $n \to \infty$		
2.3 $(x_n)^2 + 1 \to 5$	as $n \to \infty$	2.5 $\dfrac{2+x_n^2}{x_n} \to 3$	as $n \to \infty$

3. Assume that $\{x_n\}$ is a convergent sequence in $\mathbb{R}$. Prove that $\lim\limits_{n\to\infty}(x_n - x_{n+1}) = 0$.

4. If $x_n \to a$ as $n \to \infty$, prove that $x_{n+1} \to a$ as $n \to \infty$.

5. If $x_n \to +\infty$ as $n \to \infty$, prove that $x_{n+1} \to +\infty$ as $n \to \infty$.

6. Prove that $\{(-1)^n\}$ has some subsequences that converge and others that do not converge.

7. Find a convergent subsequence of $n + (-1)^{3n} n$.

8. Suppose that $\{b_n\}$ is a sequence of nonnegative numbers that converges to 0, and $\{x_n\}$ is a real sequence that satisfies $|x_n - a| \le b_n$ for large n. Prove that x_n converges to a.

9. Suppose that $\{x_n\}$ is bounded. Prove that $\dfrac{x_n}{n^k} \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$.

10. Suppose that $\{x_n\}$ and $\{y_n\}$ converge to same point. Prove that $x_n - y_n \to 0$ as $n \to \infty$

11. Prove that $x_n \to a$ as $n \to \infty$ if and only if $x_n - a \to 0$ as $n \to \infty$.

2.2 Limit theorems

Theorem 2.2.1 (**Squeeze Theorem**) *Suppose that* $\{x_n\}$, $\{y_n\}$, *and* $\{w_n\}$ *are real sequences. If* $x_n \to a$ *and* $y_n \to a$ *as* $n \to \infty$, *and there is an* $N_0 \in \mathbb{N}$ *such that*

$$x_n \le w_n \le y_n \quad \text{for all } n \ge N_0,$$

then $w_n \to a$ *as* $n \to \infty$.

Example 2.2.2 *Use the Squeeze Theorem to prove that*

$$\lim_{n\to\infty} \frac{\sin(n^2)}{2^n} = 0.$$

Theorem 2.2.3 *Let $\{x_n\}$, and $\{y_n\}$ be real sequences. If $x_n \to 0$ and $\{y_n\}$ is bounded, then*

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Example 2.2.4 *Show that* $\lim_{n\to\infty} \frac{\cos(1+n)}{n^2} = 0.$

Theorem 2.2.5 *Let $A \subseteq \mathbb{R}$.*

1. *If A has a finite supremum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \sup A \quad \text{as} \quad n \to \infty.$$

2. *If A has a finite infimum, then there is a sequence $x_n \in A$ such that*

$$x_n \to \inf A \quad \text{as} \quad n \to \infty.$$

Theorem 2.2.6 (**Additive Property**) *Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences. If $\{x_n\}$ and $\{y_n\}$ are convergent, then*

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

Theorem 2.2.7 (**Scalar Multiplicative Property**) *Let $\alpha \in \mathbb{R}$. If $\{x_n\}$ is a convergent sequence, then*

$$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n.$$

Theorem 2.2.8 (**Multiplicative Property**) *Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. Then*

$$\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

Theorem 2.2.9 (**Reciprocal Property**) *Suppose that* $\{x_n\}$ *is a convergent sequence.*

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim\limits_{n\to\infty} x_n}$$

where $\lim\limits_{n\to\infty} x_n \neq 0$ *and* $x_n \neq 0$.

Theorem 2.2.10 (**Quotient Property**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. Then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

where $\lim\limits_{n\to\infty} y_n \neq 0$ *and* $y_n \neq 0$.

Example 2.2.11 *Find the limit* $\lim_{n\to\infty} \frac{n^2+n-3}{1+3n^2}$.

Theorem 2.2.12 (**Comparison Theorem**) *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are convergent sequences. If there is an* $N_0 \in \mathbb{N}$ *such that*

$$x_n \le y_n \quad \text{for all } n \ge N_0,$$

then

$$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a,b]$ *converges to some point* c*, then* c *must belong to* $[a,b]$.

DIVERGENT.

Definition 2.2.13 *Let $\{x_n\}$ be a sequence of real numbers.*

1. *$\{x_n\}$ is said to be **diverge** to $+\infty$, written $x_n \to +\infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n > M.$$

2. *$\{x_n\}$ is said to be **diverge** to $-\infty$, written $x_n \to -\infty$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = -\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that*

$$n \geq N \quad \textit{implies} \quad x_n < M.$$

Example 2.2.14 *Show that* $\lim\limits_{n\to\infty} n = +\infty$

Example 2.2.15 *Prove that* $\lim\limits_{n\to\infty} \dfrac{n^2}{1+n} = +\infty.$

Example 2.2.16 *Prove that* $\lim_{n\to\infty} \frac{4n^2}{1-2n} = -\infty$.

Example 2.2.17 *Suppose that* $\{x_n\}$ *is a real sequence such that* $x_n \to +\infty$ *as* $n \to \infty$. *If* $x_n \neq 0$, *prove that*

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Theorem 2.2.18 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be a real sequence and* $x_n \neq 0$. *If* $\{y_n\}$ *is bounded and* $x_n \to +\infty$ *or* $x_n \to -\infty$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Example 2.2.19 *Show that* $\dfrac{\sin n}{n} \to 0$ *as* $n \to \infty$.

Theorem 2.2.20 *Let $\{x_n\}$ be a real sequence and $\alpha > 0$.*

1. *If $x_n \to +\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = +\infty$.*
2. *If $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = -\infty$.*

Theorem 2.2.21 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be real sequences. Suppose that* $\{y_n\}$ *is bounded below and* $x_n \to +\infty$ *as* $n \to \infty$. *Then*

$$\lim_{n\to\infty}(x_n + y_n) = +\infty.$$

Theorem 2.2.22 *Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that*

$$y_n > K \text{ for some } K > 0 \text{ and all } n \in \mathbb{N}.$$

It follows that

1. *if $x_n \to +\infty$ as $n \to \infty$, then $\lim_{n\to\infty}(x_n y_n) = +\infty$*
2. *if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty}(x_n y_n) = -\infty$*

Exercises 2.2

1. Prove that each of the following sequences coverges to zero.

   1.1 $x_n = \dfrac{\sin(n^4+n+1)}{n}$

   1.2 $x_n = \dfrac{n}{n^2+1}$

   1.3 $x_n = \dfrac{\sqrt{n}+1}{n+1}$

   1.4 $x_n = \dfrac{n}{2^n}$

   1.5 $x_n = \dfrac{(-1)^n}{n}$

   1.6 $x_n = \dfrac{1+(-1)^n}{2^n}$

2. Find the limit (if it exists) of each of the following sequences.

   2.1 $x_n = \dfrac{2n(n+1)}{n^2+1}$

   2.2 $x_n = \dfrac{1+n-3n^2}{3-2n+n^2}$

   2.3 $x_n = \dfrac{n^3+n+5}{5n^3+n-1}$

   2.4 $x_n = \dfrac{\sqrt{2n^2-1}}{n+1}$

   2.5 $x_n = \sqrt{n+2}-\sqrt{n}$

   2.6 $x_n = \sqrt{n^2+n}-n$

3. Prove that each of the following sequences coverges to $-\infty$ or $+\infty$.

   3.1 $x_n = n^2$

   3.2 $x_n = -n$

   3.3 $x_n = \dfrac{n}{1+\sqrt{n}}$

   3.4 $x_n = \dfrac{n^2+1}{n+1}$

   3.5 $x_n = \dfrac{1-n^2}{n}$

   3.6 $x_n = \dfrac{2^n}{n}$

4. Let $A \subseteq \mathbb{R}$. If A has a finite supremum, then there is a sequence $x_n \in A$ such that

$$x_n \to \sup A \quad \text{as} \quad n \to \infty.$$

5. Prove that given $x \in \mathbb{R}$ there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$.

6. Use the result Excercise 1.2, show that the following

   6.1 Suppose that $0 \le x_1 \le 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n \in \mathbb{N}$.

   If $x_n \to x$ as $n \to \infty$, prove that $x = 0$ or 1.

6.2 Suppose that $x_1 > 0$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$.

If $x_n \to x$ as $n \to \infty$, prove that $x = 2$.

7. Let $\{x_n\}$ be a real sequence and $\alpha > 0$. If $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (\alpha x_n) = -\infty$.

8. Let $\{x_n\}$ and $\{y_n\}$ be real sequences such that $y_n > K$ for some $K > 0$ and all $n \in \mathbb{N}$. Prove that if $x_n \to -\infty$ as $n \to \infty$, then $\lim_{n\to\infty} (x_n y_n) = -\infty$.

9. Let $\{x_n\}$ and $\{y_n\}$ are real sequences. Suuppose that $\{y_n\}$ is bounded above and $x_n \to -\infty$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} (x_n + y_n) = -\infty.$$

10. Interpret a decimal expansion $0.a_1a_2a_3...$ as

$$0.a_1a_2a_3... = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{a_k}{10^k}.$$

Prove that

10.1 $0.5 = 0.4999...$

10.2 $1 = 0.999...$

2.3 Bolzano-Weierstrass Theorem

MONOTONE.

Definition 2.3.1 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.*

1. *$\{x_n\}$ is said to be* ***increasing*** *if and only if $x_1 \leq x_2 \leq x_3 \leq \ldots$ or*

$$x_n \leq x_{n+1} \quad \textit{for all } n \in \mathbb{N}.$$

2. *$\{x_n\}$ is said to be* ***decreasing*** *if and only if $x_1 \geq x_2 \geq x_3 \geq \ldots$ or*

$$x_n \geq x_{n+1} \quad \textit{for all } n \in \mathbb{N}.$$

3. *$\{x_n\}$ is said to be* ***monotone*** *if and only if it is either increasing or decreasing.*

If $\{x_n\}$ is increasing and converges to a, we shall write $x_n \uparrow a$ as $n \to \infty$.

If $\{x_n\}$ is decreasing and converges to a, we shall write $x_n \downarrow a$ as $n \to \infty$.

Example 2.3.2 *Determine whether $\{x_n\}_{n\in\mathbb{N}}$ is increasing or decreasing or NOT both.*

Sequences	Decreasing	Increasing	Monotone
$\{n\}_{n\in\mathbb{N}}$			
$\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$			
$\{1\}_{n\in\mathbb{N}}$			
$\{(-1)^n\}_{n\in\mathbb{N}}$			

Theorem 2.3.3 (**Monotone Converegence Theorem (MCT)**) *If $\{x_n\}$ is increasing and bounded above, or if it is decreasing and bounded below, then $\{x_n\}$ has a finite limit.*

Theorem 2.3.4 *If* $|a| < 1$, *then* $a^n \to 0$ *as* $n \to \infty$.

Example 2.3.5 *Find the limit of* $\left\{ \dfrac{3^{n+1}+1}{3^n+2^n} \right\}$.

Definition 2.3.6 *A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be* ***nested*** *if and only if*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \quad \text{or} \quad I_{n+1} \subseteq I_n \text{ for all } n \in \mathbb{N}.$$

Example 2.3.7 *Show that $I_n = [\frac{1}{n}, 1]$ is nested.*

Theorem 2.3.8 (**Nested Interval Property**) *If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then*

$$E = \bigcap_{n\in\mathbb{N}} I_n := \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

contains at least one number. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E contains exactly one number.

Theorem 2.3.9 (**Bolzano-Weierstrass Theorem**) *Every bounded sequence of real numbers has a convergence subsequence.*

Exercises 2.3

1. Prove that
$$x_n = \frac{(n^2+22n+65)\sin(n^3)}{n^2+n+1}$$
has a convergence sunsequence.

2. If $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ has a finite limit.

3. Suppose that $E \subset \mathbb{R}$ is nonempty bpunded set and $\sup E \notin E$. Prove that there exist a strictly increasing sequence $\{x_n\}$ $(x_1 < x_2 < x_3 < ...)$ that converges to $\sup E$ such that $x_n \in E$ for all $n \in \mathbb{N}$.

4. Suppose that $\{x_n\}$ is a monotone increasing in $\mathbb{R}$ (not necessarily bounded above). Prove that there is extended real number x such that $x_n \to x$ as $n \to \infty$.

5. Suppose that $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Prove that
$$x_n \downarrow 0 \text{ as } n \to \infty \quad \text{and} \quad \frac{x_{n+1}}{x_n} \to \frac{1}{2}, \text{ as } n \to \infty$$

6. If $a > 0$, prove that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$. Use the resulte to find the limit of $\{3^{\frac{n+1}{n}}\}$.

7. Let $0 \le x_1 \le 3$ and $x_{n+1} = \sqrt{2x_n + 3}$ for $n \in \mathbb{N}$. Prove that $x_n \uparrow 3$ as $n \to \infty$.

8. Suppose that $x_1 \ge 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Prove that $x_n \downarrow 2$ as $n \to \infty$. What happens when $1 \le x_1 < 2$?

9. Prove that
$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

10. Suppose that $x_0 \in \mathbb{R}$ and $x_n = \dfrac{1 + x_{n-1}}{2}$ for $n \in \mathbb{N}$. Prove that $x_n \to 1$ as $n \to \infty$.

11. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove that

    11.1 if $x_n \downarrow 0$, then $x_n > 0$ for all $n \in \mathbb{N}$.

11.2 if $x_n \uparrow 0$, then $x_n < 0$ for all $n \in \mathbb{N}$.

12. Let $0 < y_1 < x_1$ and set

$$x_{n+1} = \frac{x_n + y_n}{2} \quad \text{and} \quad y_{n+1} = \sqrt{x_n y_n}, \quad \text{for } n \in \mathbb{N}$$

12.1 Prove that $0 < y_n < x_n$ for all $n \in \mathbb{N}$.

12.2 Prove that y_n is increasing and bounded above, and x_n is decreasing and bounded below.

12.3 Prove that $0 < x_{n+1} - y_{n+1} < \dfrac{x_1 - y_1}{2^n}$ for $n \in \mathbb{N}$

12.4 Prove that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$. (the common value is called the arithemetic-geometric mean of x_1 and y_1.)

13. Suppose that $x_0 = 1, y_0 = 0$

$$x_n = x_{n-1} + 2y_{n-1},$$

and

$$y_n = x_{n-1} + y_{n-1}$$

for $n \in \mathbb{N}$. Prove that $x_n^2 - 2y_n^2 = \pm 1$ for $n \in \mathbb{N}$ and

$$\frac{x_n}{y_n} \to \sqrt{2} \quad \text{as} \quad n \to \infty.$$

14. **(Archimedes)** Suppose that $x_0 = 2\sqrt{3}, y_0 = 3$,

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}}, \quad \text{and} \quad y_n = \sqrt{x_n y_{n-1}} \quad \text{for } n \in \mathbb{N}.$$

14.1 Prove that $x_n \downarrow x$ and $y_n \uparrow y$, as $n \to \infty$, for some $x, y \in \mathbb{R}$.

14.2 Prove that $x = y$ and

$$3.14155 < x < 3.14161.$$

(The actual value of x is π.)

2.4 Cauchy sequences

Definition 2.4.1 *A sequence of points $x_n \in \mathbb{R}$ is said to be **Cauchy** if and only if every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$n, m \geq N \quad \textit{imply} \quad |x_n - x_m| < \varepsilon.$$

Example 2.4.2 *Show that $\left\{\frac{1}{n}\right\}$ is Cauchy.*

Example 2.4.3 *Show that $\left\{\frac{n}{n+1}\right\}$ is Cauchy.*

Theorem 2.4.4 *The sum of two Cauchy sequences is Cauchy.*

Theorem 2.4.5 *If* $\{x_n\}$ *is convergent, then* $\{x_n\}$ *is Cauchy.*

Theorem 2.4.6 (**Cauchy's Theorem**) *Let* $\{x_n\}$ *be a sequence of real numbers. Then*

$$\{x_n\} \text{ is Cauchy if and only if } \{x_n\} \text{ converges to some point in } \mathbb{R}.$$

Example 2.4.7 *Prove that any real sequence* $\{x_n\}$ *that satisfies*

$$|x_n - x_{n+1}| \leq \frac{1}{2^n}, \quad n \in \mathbb{N},$$

is convergent.

Exercises 2.4

1. Use definition to show that $\{x_n\}$ is Cauchy if

   1.1 $x_n = \dfrac{1}{n^2}$

   1.2 $x_n = \dfrac{n}{n+1}$

2. Prove that the product of two Cauchy sequences is Cauchy.

3. Prove that if $\{x_n\}$ is a sequence that satisfies

$$|x_n| \leq \frac{1+n}{1+n+2n^2}$$

   for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.

4. Suppose that $x_n \in \mathbb{N}$ for $n \in \mathbb{N}$. If $\{x_n\}$ is Cauchy prove that there are numbers a and N such that $x_n = a$ for all $n \geq N$.

5. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that there is an $N \in \mathbb{N}$ satisfying the statement:

$$\text{if } n, m \geq N, \text{ then } \quad |x_n - x_m| < \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

   Prove that $\{a_n\}$ converges.

   $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} x_k$ exists and is finite.

6. Let $\{x_n\}$ be Cauchy. Prove that $\{x_n\}$ converges if and only if at least one of its subsequence converges.

7. Prove that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{(-1)^k}{k}$ exists and is finite.

8. Let $\{x_n\}$ be a sequence. Suppose that there is an $a > 1$ such that

$$|x_{k+1} - x_k| \leq a^{-k}$$

   for all $k \in \mathbb{N}$. Prove that $x_n \to x$ for some $x \in \mathbb{R}$.

9. Show that a sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Chapter 3

Topology on $\mathbb{R}$

3.1 Open sets

Open sets are among the most important subsets of $\mathbb{R}$. A collection of open sets is called a topology, and any property (such as convergence, compactness, or continuity) that can be dened entirely in terms of open sets is called a **topological property**.

Definition 3.1.1 *A set $E \subseteq \mathbb{R}$ is **open** if for every $x \in E$ there exists a $\delta > 0$ such that*

$$(x-\delta, x+\delta) \subseteq E.$$

In other word,

$$\begin{aligned} E \textit{ is open} \quad &\leftrightarrow \quad \forall x \in E\ \exists \delta > 0,\ (x-\delta, x+\delta) \subseteq E \\ &\textit{and} \\ E \textit{ is not open} \quad &\leftrightarrow \quad \exists x \in E\ \forall \delta > 0,\ (x-\delta, x+\delta) \not\subseteq E. \end{aligned}$$

Since the empty set has no element, by definition it imples that $\varnothing$ is open. For $E = \mathbb{R}$, we obatin

$$\forall x \in \mathbb{R}\ \exists \delta > 0,\ (x-\delta, x+\delta) \subseteq \mathbb{R} \text{ is true.}$$

It follows that R is open.

Example 3.1.2 *Show that interval* $(0, 1)$ *is open.*

Theorem 3.1.3 *Intervals* (a, b)*,* (a, ∞) *and* $(-\infty, b)$ *are open.*

Example 3.1.4 *Show that* $[0, 1)$ *is not open.*

Theorem 3.1.5 *Let A and B be open. Prove that $A \cup B$ and $A \cap B$ are open.*

Theorem 3.1.6 *Let $A_1, A_2, .., A_n$ be open sets. Then*

1. $\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is open.*
2. $\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is open.*

NEIGHBORHOOD.

Next, we introduce the notion of the neighborhood of a point, which often gives clearer, but equivalent, descriptions of topological concepts than ones that use open intervals.

Definition 3.1.7 *A set $U \subseteq \mathbb{R}$ is a* ***neighborhood*** *of a point $x \in \mathbb{R}$ if*

$$(x - \delta, x + \delta) \subseteq U \quad \text{for some } \delta > 0.$$

For example $x = 1$, we have $(0, 2)$, $[0, 2]$ and $[0, 2)$ to be neighborhoods of 1.

Theorem 3.1.8 *A set $E \subseteq \mathbb{R}$ is open if every $x \in E$ has a neighborhood U such that $U \subseteq E$.*

Theorem 3.1.9 *A sequence $\{x_n\}$ of real numbers converges to a limit $x \in \mathbb{R}$ if and only if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.*

Exercises 3.1

1. Show that interval $[a, b]$, $[a, b)$ and $(a, b]$, are not open.

2. Show that interval $[a, \infty)$ and $(-\infty, b]$ are not open.

3. Give two neighborhoods of $x = 2$.

4. Let A and B be subsets of $\mathbb{R}$. Suppose that A and B are open.
   Determine whether $A \backslash B$ is open.

5. Let $U \subseteq \mathbb{R}$ be a nonempty open set. Show that $\sup U \notin U$ and $\inf U \notin U$.

6. Let $A_1, A_2, .., A_n$ be open sets. Prove that

   6.1 $\displaystyle\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ is open.

   6.2 $\displaystyle\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ is open.

7. Find a sequence I_n of bounded, and open interval that

$$I_{n+1} \subset I_n \text{ for each } n \in \mathbb{N} \text{ and } \bigcap_{n=1}^{\infty} I_n = \varnothing.$$

3.2 Closed sets

Definition 3.2.1 *A set $F \subseteq \mathbb{R}$ is* ***closed*** *if*

$$F^c = \mathbb{R} \backslash F = \{x \in \mathbb{R} : x \notin F\} \text{ is open.}$$

Since $\varnothing^c = \mathbb{R}$ and $\mathbb{R}^c = \varnothing$ ($\varnothing$ and $\mathbb{R}$ are open), $\varnothing$ and $\mathbb{R}$ are closed sets.

Example 3.2.2 *Show that interval $[0,1]$ is closed.*

Example 3.2.3 *Show that $[0,1)$ is neither open nor closed.*

Theorem 3.2.4 *Let A and B be closed. Prove that $A \cup B$ and $A \cap B$ are closed.*

Theorem 3.2.5 *Let $A_1, A_2, .., A_n$ be closed sets. Then*

1. $\bigcup_{k=1}^{n} A_k := A_1 \cup A_2 \cup ... \cup A_n$ *is closed.*

2. $\bigcap_{k=1}^{n} A_k := A_1 \cap A_2 \cap ... \cap A_n$ *is closed.*

Exercises 3.2

1. Show that interval $[a,b]$, $[a,\infty)$ and $(-\infty,b]$ are closed.

2. The set of rational numbers $\mathbb{Q}\subset\mathbb{R}$ is neither open nor closed.

3. Show that every closed interval I is a closed set.

4. Is $\displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right)$ open or closed ?

5. Is $\displaystyle\bigcup_{n=1}^{\infty}\left[\frac{1}{n},\frac{n-1}{n}\right]$ open or closed ?

6. Suppose, for $n\in\mathbb{N}$, the intervals $I_n=[a_n,b_n]$ are such that $I_{n+1}\subset I_n$. If

$$a=\sup\{a_n:n\in\mathbb{N}\}\quad\text{and}\quad b=\inf\{b_n:n\in\mathbb{N}\},$$

show that $\displaystyle\bigcap_{n=1}^{\infty}I_n=[a,b]$.

7. Find a sequence I_n of closed interval that $I_{n+1}\subset I_n$ for each $n\in\mathbb{N}$ and $\displaystyle\bigcap_{n=1}^{\infty}I_n=\varnothing$.

8. Suppose that $U\subseteq\mathbb{R}$ is a nonempty open set. For each $x\in U$, let

$$J_x=\big(x-\varepsilon,x+\delta\big),$$

where the union is taken over all $\varepsilon>0$ and $\delta>0$ such that $(x-\varepsilon,x+\delta)\subset U$.

8.1 Show that for every $x,y\in U$, either $J_x\cap J_y=\varnothing$, or $J_x=J_y$.

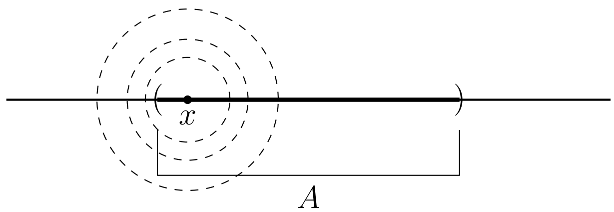
8.2 Show that $U=\displaystyle\bigcup_{x\in B}J_x$, where $B\subseteq U$ is either finite or countable.

3.3 Limit points

Definition 3.3.1 *A point $x \in \mathbb{R}$ is called a **limit point** of a set $A \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there exists $a \in A$, $a \neq x$, such that $a \in (x-\varepsilon, x+\varepsilon)$ or*

$$[(x-\varepsilon, x) \cup (x, x+\varepsilon)] \cap A \neq \varnothing.$$

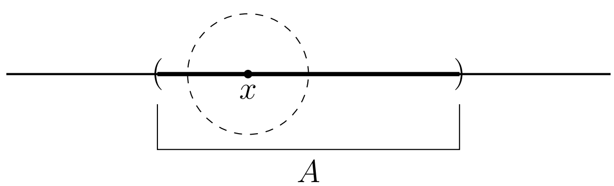
We denote the set of all limit points of a set A by A'.



Definition 3.3.2 *Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is an **interior point** of A if there exists an $\delta > 0$ such that*

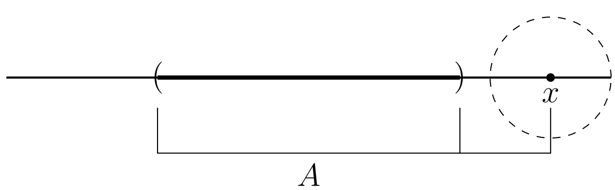
$$(x-\delta, x+\delta) \subseteq A.$$

The set of all interior points of A is called the interior of A, denoted A°.



Definition 3.3.3 *Suppose $A \subseteq \mathbb{R}$. A point $x \in A$ is called an **isolated point** of A if there exists an $\delta > 0$ such that*

$$A \cap (x-\delta, x+\delta) = \{x\}.$$



Example 3.3.4 *Fill the blanks of the following table.*

Set	Set of limit points	Set of interior points	Set of isolated points
$[0, 1]$			
$(0, 1)$			
$[0, 1)$			
$(0, 1] \cup \{3\}$			
$\{1\}$			
$\mathbb{N}$			
$\mathbb{Q}$			

Example 3.3.5 *Show that* 0 *is a limit point of* $(0, 1)$.

Theorem 3.3.6 *Let A and B be sets. If $A \subseteq B$, then $A' \subseteq B'$.*

Theorem 3.3.7 *Let A be a closed subset of $\mathbb{R}$. Then $A' \subseteq A$.*

CLOSURE.

Definition 3.3.8 *Given a set $A \subseteq R$, the set $\bar{A} = A \cup A'$ is called the* ***closure*** *of A.*

Example 3.3.9 *Fill the blanks of the following table.*

Set	Set of limit points	Closure
$[0,1]$		
$(0,1)$		
$[0,1)$		
$(0,1] \cup \{3\}$		
$\{1\}$		
$\mathbb{N}$		
$\mathbb{Q}$		

Theorem 3.3.10 *Let A and B be subsets of $\mathbb{R}$. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Theorem 3.3.11 *Let $A \subseteq \mathbb{R}$. Then $\bar{A}$ is closed.*

Theorem 3.3.12 *Let $A \subseteq \mathbb{R}$. Then A is closed if and only if $A = \bar{A}$.*

Theorem 3.3.13 *A set $F \subseteq \mathbb{R}$ is closed if and only if*

the limit of every convergent sequence in F belongs to F.

Exercises 3.3

1. Identify the limit points, interior point and isolated points of the following sets:

   1.1 $A = (0, 1) \cup \{3\}$

   1.2 $A = [0, 1]^c$

   1.3 $A = [1, \infty)$

   1.4 $A = (0, 1) \cup [3, 4]$

   1.5 $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

   1.6 $A = [0, 1] \cap \mathbb{Q}$

2. Find A', A° and $\bar{A}$ where

   2.1 $A = (0, 1)$

   2.2 $A = [0, 1]$

   2.3 $A = [0, \infty)$

   2.4 $A = (0, 1) \cup \{2, 3\}$

   2.5 $A = \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\}$

   2.6 $A = \mathbb{Q}$

3. Let A and B be two subset of $\mathbb{R}$. Show that $(A \cup B)' = A' \cup B'$.

4. Let A and B be two subset of $\mathbb{R}$. Determine whether

   4.1 $(A \cap B)' = A' \cap B'$

   4.2 $\overline{A \cup B} = \bar{A} \cup \bar{B}$

   4.3 $\overline{A \cap B} = \bar{A} \cap \bar{B}$

   4.4 $(A \cup B)^\circ = A^\circ \cup B^\circ$

   4.5 $(A \cap B)^\circ = A^\circ \cap B^\circ$

   4.6 if $\bar{A} \subseteq \bar{B}$, then $A \subseteq B$.

5. Prove that A° is open.

6. Prove that A is open if and only if $A = A^\circ$.

7. Suppose x is a limit point of the set A. Show that for every $\varepsilon > 0$, the set

$$(x - \varepsilon, x + \varepsilon) \cap A \text{ is infinite.}$$

8. Suppose that $A_k \subseteq \mathbb{R}$ for each $k \in \mathbb{N}$, and let $B = \bigcup_{k=1}^{\infty} A_k$. Show that $\bar{B} = \bigcup_{k=1}^{\infty} \bar{A}_k$.

9. If the limit of every convergent sequence in F belongs to $F \subseteq \mathbb{R}$, prove that F is closed.

Chapter 4

Limit of Functions

4.1 Limit of Functions

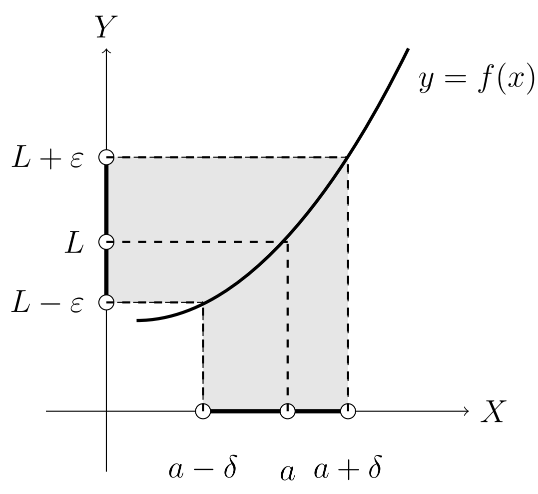
Definition 4.1.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a limit point of E. Then $f(x)$ is said to* ***converge*** *to L, as x* ***approaches*** *a, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in E$,*

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

In this case we write

$$\lim_{x \to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

and call L the ***limit*** *of $f(x)$ as x approaches a.*



Example 4.1.2 *Suppose that* $f(x) = 2x + 1$. *Prove that*

$$\lim_{x \to 1} f(x) = 3.$$

Example 4.1.3 *Let* $f(x) = \sqrt{x^2}$ *where* $x \in \mathbb{R}$. *Prove that* $f(x) \to 0$ *as* $x \to 0$.

Example 4.1.4 *Prove that*

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$$

Example 4.1.5 *Prove that*

$$\lim_{x\to 3} x^2 = 9.$$

Example 4.1.6 *Prove that* $f(x) = \dfrac{1}{x} \to 1$ *as* $x \to 1$.

Theorem 4.1.7 (**Limit of Constant function**) *The limit of a constant function is equal to the constant.*

Theorem 4.1.8 (**Limit of Linear function**) *Let m and c be constant such that $f(x) = mx + c$ for all $x \in \mathbb{R}$. Then*

$$\lim_{x \to a}(mx + c) = ma + c.$$

Theorem 4.1.9 *Let $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ be a limit point of E. If*

$$f(x) = g(x) \text{ for all } x \in E \backslash \{a\} \quad \text{and} \quad f(x) \to L \text{ as } x \to a,$$

then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Example 4.1.10 *Prove that $f(x) = \dfrac{x^2 - 1}{x - 1}$ has a limit as $x \to 1$.*

Theorem 4.1.11 (**Sequential Characterization of Limit (SCL)**) *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be a limit point of E. Then*

$$\lim_{x \to a} f(x) = L \quad \textit{exists}$$

if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in E \setminus \{a\}$ that converges to a as $n \to \infty$.

Example 4.1.12 *Use the SCL to prove that*

$$f(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has no limit as $x \to 0$.

Example 4.1.13 *Use the SCL to prove that*

$$e^{-\frac{1}{x}} \to 0 \quad as \quad x \to 0^+.$$

Theorem 4.1.14 *Let $\alpha \in \mathbb{R}$, $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ be functions and let $a \in \mathbb{R}$ be a limit point of E. If $f(x)$ and $g(x)$ converge as x approaches a, then so do*

$$(f+g)(x),\ (\alpha f)(x),\ (fg)(x) \text{ and } (\tfrac{f}{g})(x).$$

In fact,

1. $\lim\limits_{x\to a}(f+g)(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}(\alpha f)(x) = \alpha \lim\limits_{x\to a} f(x)$
3. $\lim\limits_{x\to a}(fg)(x) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ *when the limit of $g(x)$ is nonzero.*

Example 4.1.15 *Show that $\lim\limits_{x\to a} x^2 = a^2$ fo all $a \in \mathbb{R}$.*

Theorem 4.1.16 *Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is a function. Let $a \in \mathbb{R}$ be a limit point of E. Then,*

$$\lim_{x \to a} |f(x)| = 0 \quad \text{if and only if} \quad \lim_{x \to a} f(x) = 0.$$

Theorem 4.1.17 (**Squeeze Theorem for Functions**) *Suppose that $E \subseteq \mathbb{R}$ and $f, g, h : E \to \mathbb{R}$ are functions. Let $a \in \mathbb{R}$ be a limit point of E. If*

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in E \backslash \{a\},$$

and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then the limit of $f(x)$ exists, as $x \to a$ and

$$\lim_{x \to a} f(x) = L.$$

Corollary 4.1.18 *Suppose that $E \subseteq \mathbb{R}$ and $f, g : E \to \mathbb{R}$ are functions. Let $a \in \mathbb{R}$ be a limit point of E and $M > 0$. If*

$$|g(x)| \leq M \quad \text{for all } x \in E\backslash\{a\} \quad \text{and} \quad \lim_{x\to a} f(x) = 0,$$

then

$$\lim_{x\to a} f(x)g(x) = 0.$$

Example 4.1.19 *Show that* $\displaystyle\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

Theorem 4.1.20 (**Comparision Theorem for Functions**) *Suppose that* $E \subseteq \mathbb{R}$ *and* $f, g : E \to \mathbb{R}$ *are functions. Let* $a \in \mathbb{R}$ *be a limit point of* E*. If* f *and* g *have a limit as* x *approaches* a *and*

$$f(x) \leq g(x), \quad x \in E \backslash \{a\},$$

then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Exercises 4.1

1. Use Definition 4.1.1, prove that each of the following limit exists.

   1.1 $\lim_{x \to 1} x^2 = 1$

   1.2 $\lim_{x \to 2} x^2 - x + 1 = 3$

   1.3 $\lim_{x \to -1} x^3 + 1 = 0.$

   1.4 $\lim_{x \to 0} \frac{x-1}{x+1} = -1$

2. Decide which of the following limit exist and which do not.

   2.1 $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

   2.2 $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

   2.3 $\lim_{x \to 0} \tan\left(\frac{1}{x}\right)$

3. Evaluate the following limit using result from this section.

   3.1 $\lim_{x \to 1} \frac{x^2 + x - 2}{x^3 - x}$

   3.2 $\lim_{x \to \sqrt{\pi}} \frac{\sqrt[3]{\pi - x^2}}{x + \pi}$

   3.3 $\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$

   3.4 $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

4. Prove that $\lim_{x \to 0} x^n \sin\left(\frac{1}{x}\right)$ exists for all $n \in \mathbb{N}$.

5. Show that $\lim_{x \to a} x^n = a^n$ fo all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

6. Prove that $\lim_{x \to a} |f(x)| = 0$ if and only if $\lim_{x \to a} f(x) = 0$.

7. Prove Squeeze Theorem for Functions.

8. Prove Comparision Theorem for Functions.

9. Suppose that f is a real function.

   9.1 Prove that if

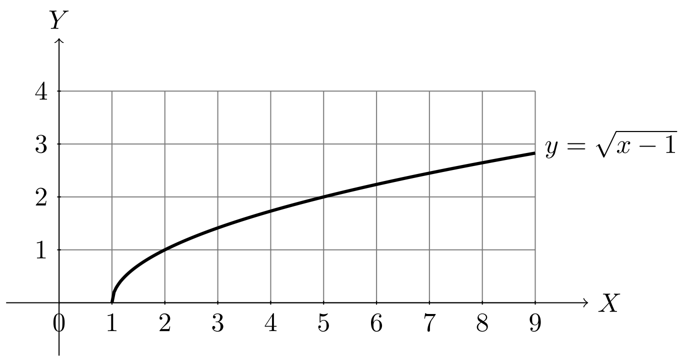
   $$\lim_{x \to a} f(x) = L$$

   exists, then $|f(x)| \to |L|$ as $x \to a$.

   9.2 Show that there is a function such that as $x \to a$, $|f(x)| \to |L|$ but the limit of $f(x)$ does not exist.

4.2 One-sided limit

What is the limit of $f(x) := \sqrt{x-1}$ as $x \to 1$.



A reasonable answer is that the limit is zero. This function, however, does not satisfy Definition 4.1.1 because it is not defined on an OPEN interval containg $a = 1$. Indeed, f is defined only for $x \geq 1$. To handle such situations, we introduce one-sided limits.

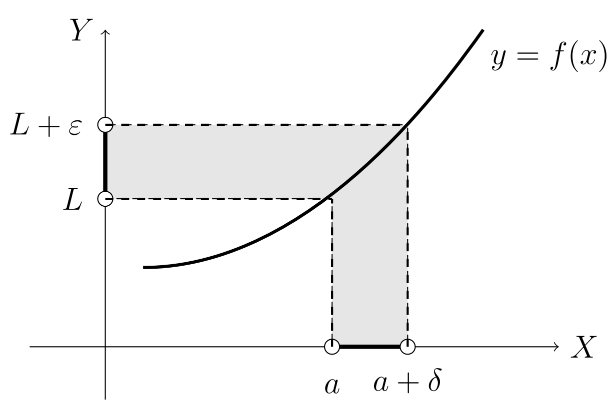
Definition 4.2.1 *Let $a \in \mathbb{R}$.*

1. *A real function f said to **converge** to L as x **approaches** a **from the right** if and only if f defined on some interval I with left endpoint a and every $\varepsilon > 0$ there is a $\delta > 0$ such that $a + \delta \in I$ and for all $x \in I$,*

$$a < x < a + \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

*In this case we call L the **right-hand limit** of f at a, and denote it by*

$$f(a^+) := L =: \lim_{x \to a^+} f(x).$$

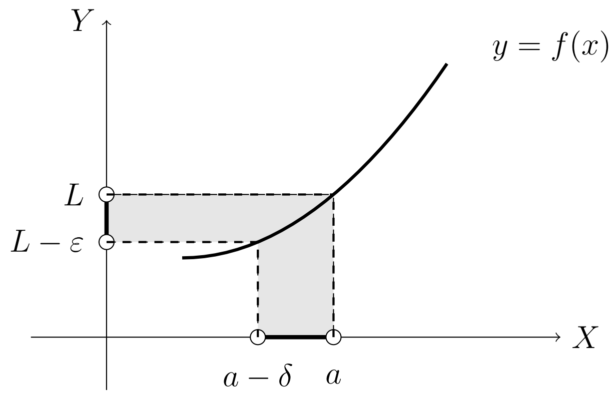


2. *A real function f said to **converge** to L as x **approaches** a **from the left** if and only if f defined on some interval I with right endpoint a and every $\varepsilon > 0$ there is a $\delta > 0$ such that $a + \delta \in I$ and for all $x \in I$,*

$$a - \delta < x < a \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

*In this case we call L the **left-hand limit** of f at a, and denote it by*

$$f(a^-) := L =: \lim_{x \to a^-} f(x).$$



Example 4.2.2 *Prove that*

1. $\lim_{x \to 1^+} \sqrt{x - 1} = 0$
2. $\lim_{x \to 0^-} \sqrt{-x} = 0$

Example 4.2.3 *If* $f(x) = \dfrac{|x|}{x}$, *prove that* f *has one-sided limit at* $a = 0$ *but* $\lim_{x \to 0} f(x) = 0$ *DNE.*

Theorem 4.2.4 *Let f be a real function. Then the limit*

$$\lim_{x \to a} f(x)$$

exists and equals to L if and only if

$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Example 4.2.5 *Use Theorem 4.2.4 to show that* $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ 2x+1 & \text{if } x < 0 \end{cases}$ *has limit at* $a = 0$.

Exercises 4.2

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1. Use definitons to prove that $\lim_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = 2x^2 + 1,$ $\quad a = 1$, and $L = 3$.

   1.2 $f(x) = \dfrac{x-1}{|1-x|},$ $\quad a = 1$, and $L = 1$.

   1.3 $f(x) = \sqrt{3x-5},$ $\quad a = 2$, and $L = 1$.

2. Use definitons to rove that $\lim_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = 1 + x^2,$ $\quad a = 1$, and $L = 2$.

   2.2 $f(x) = \sqrt{1-x^2},$ $\quad a = 1$, and $L = 0$.

   2.3 $f(x) = \dfrac{1-x^2}{1+x},$ $\quad a = 1$, and $L = 0$.

3. Evauate the following limit when they exist.

   3.1 $\lim_{x\to 0^+} \dfrac{x+1}{x^2-2}$

   3.2 $\lim_{x\to 1^-} \dfrac{x^3-3x+2}{x^3-1}$

   3.3 $\lim_{x\to \pi^+} (x^2+1)\sin x$

   3.4 $\lim_{x\to \frac{\pi}{2}^-} \dfrac{\cos x}{1-\sin x}$

4. Prove that $\dfrac{\sqrt{1-\cos x}}{\sin x} \to \dfrac{\sqrt{2}}{2}$ as $x \to 0^+$.

5. Determine whether the following functions are limit at a.

   5.1 $f(x) = \begin{cases} 3x+1 & \text{if } x \geq 1 \\ x+3 & \text{if } x < 1 \end{cases}$ and $\quad a = 1$

   5.2 $f(x) = \begin{cases} 2-2x & \text{if } x \geq 0 \\ \sqrt{1-x} & \text{if } x < 0 \end{cases}$ and $\quad a = 0$

6. Suppose that $f : [0,1] \to \mathbb{R}$ and $f(a) = \lim_{x\to a} f(x)$ for all $x \in [0,1]$. Prove that

$$f(q) = 0 \text{ for all } q \in \mathbb{Q} \cap [0,1] \text{ if and only if } f(x) = 0 \text{ for all } x \in [0,1].$$

4.3 Infinite limit

The definition of limit of real functions can be expanded to include extended real numbers.

Definition 4.3.1 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to L$ as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x > M \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

   In this case we shall write $\lim_{x\to\infty} f(x) = L$.

2. *We say that $f(x) \to L$ as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(-\infty, -c) \subseteq E$ and for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that*

$$x < M \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

   In this case we shall write $\lim_{x\to-\infty} f(x) = L$.

Example 4.3.2 *Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$.*

Example 4.3.3 *Prove that* $\lim_{x\to\infty} \dfrac{x-1}{x+1}$ *exists and equals to 1.*

Example 4.3.4 *Prove that* $\lim_{x\to\infty} \dfrac{1}{x^2+1} = 0.$

Example 4.3.5 *Prove that* $\lim_{x \to -\infty} \frac{1}{x} = 0.$

Example 4.3.6 *Prove that* $\lim_{x \to -\infty} \frac{x}{x+1} = 1.$

Definition 4.3.7 *Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ be a function.*

1. *We say that $f(x) \to +\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M > 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \textit{implies} \quad f(x) > M.$$

   In this case we shall write $\lim_{x \to a} f(x) = +\infty$.

2. *We say that $f(x) \to -\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \backslash \{a\} \subset E$ and for every $M < 0$ there is a $\delta > 0$ such that*

$$0 < |x - a| < \delta \quad \textit{implies} \quad f(x) < M.$$

   In this case we shall write $\lim_{x \to a} f(x) = -\infty$.

Obviousl modification define $f(x) \to \pm\infty$ as $x \to a^+$ and $x \to a^-$, and $f(x) \to \pm\infty$ as $x \to \pm\infty$.

Example 4.3.8 *Prove that* $\lim_{x \to 0} \frac{1}{|x|} = +\infty$.

Example 4.3.9 *Prove that* $\displaystyle\lim_{x\to 1^+} \frac{x}{1-x} = -\infty.$

Example 4.3.10 *Prove that* $\displaystyle\lim_{x\to 1^-} \frac{x}{1-x} = +\infty.$

Exercises 4.3

1. Use definitons to prove that $\lim_{x\to a^+} f(x)$ exists and equal to L in each of the following cases.

   1.1 $f(x) = \dfrac{1}{x-3}$, $a = 3$, and $L = +\infty$.

   1.2 $f(x) = -\dfrac{1}{x}$, $a = 0$, and $L = -\infty$.

2. Use definitons to prove that $\lim_{x\to a^-} f(x)$ exists and equal to L in each of the following cases.

   2.1 $f(x) = \dfrac{x}{x^2-4}$, $a = 2$, and $L = -\infty$.

   2.2 $f(x) = \dfrac{1}{1-x^2}$, $a = 1$, and $L = +\infty$.

3. Use definition to prove that the follwing limits

   3.1 $\lim_{x\to\infty} \dfrac{2x+1}{x+1} = 2$

   3.2 $\lim_{x\to-\infty} \dfrac{1-x}{2x+1} = -\dfrac{1}{2}$

   3.3 $\lim_{x\to\infty} \dfrac{2x^2+1}{1-x^2} = -2$

   3.4 $\lim_{x\to 2} \dfrac{x}{|x-2|} = +\infty$

   3.5 $\lim_{x\to 2^+} \dfrac{x+1}{x-2} = +\infty$

   3.6 $\lim_{x\to 2^-} \dfrac{x+1}{x-2} = -\infty$

4. Evauate the following limit when they exist.

   4.1 $\lim_{x\to\infty} \dfrac{3x^2-13x+4}{1-x-x^2}$

   4.2 $\lim_{x\to\infty} \dfrac{x^2+x+2}{x^3-x-2}$

   4.3 $\lim_{x\to-\infty} \dfrac{x^3-1}{x^2+2}$

   4.4 $\lim_{x\to\infty} \arctan x$

   4.5 $\lim_{x\to\infty} \dfrac{\sin x}{x^2}$

   4.6 $\lim_{x\to-\infty} x^2 \sin x$

5. Prove that $\dfrac{\sin(x+3)-\sin 3}{x}$ converges to 0 as $x \to \infty$.

6. Prove the following comparision theorems for real functions.

   6.1 If $f(x) \geq g(x)$ and $g(x) \to \infty$ as $x \to a$, then $f(x) \to \infty$ as $x \to a$.

   6.2 If $f(x) \leq g(x) \leq h(x)$ and $L = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x)$, then $g(x) \to L$ as $x \to \infty$.

7. Recall that a **polynomial of degree n** is a functon of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_j \in \mathbb{R}$ for $j = 0, 1, ..., n$ and $a_n \neq 0$.

7.1 Prove that $\lim_{x\to a} x^n = a^n$ for $n = 0, 1, 2, ...$

7.2 Prove that if P is a polynomial, then

$$\lim_{x\to a} P(x) = P(a)$$

for every $a \in \mathbb{R}$.

7.3 Suppose that P is a polynomial and $P(a) > 0$. Prove that $\dfrac{P(x)}{x-a} \to \infty$ as $x \to a^+$, $\dfrac{P(x)}{x-a} \to -\infty$ as $x \to a^-$, but

$$\lim_{x\to a} \frac{P(x)}{x-a}$$

does not exist.

8. **Cauchy**. Suppose that $f : \mathbb{N} \to \mathbb{R}$. If

$$\lim_{n\to\infty} f(n+1) - f(n) = L,$$

prove that $\lim_{n\to\infty} \dfrac{f(n)}{n}$ exists and equals L.

Chapter 5

Continuity on $\mathbb{R}$

5.1 Continuity

Definition 5.1.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*

f is said to be ***continuous at a point*** *$a \in E$ if and only if given $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \text{ and } x \in E \quad \text{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.1.2 *Let $f(x) = 2x - 1$ where $x \in \mathbb{R}$. Prove that f is continuous at $x = 1$.*

Example 5.1.3 *Let* $f(x) = x^2$ *where* $x \in \mathbb{R}$. *Prove that* f *is continuous at* $x = 2$.

Example 5.1.4 *Let* $f(x) = \sqrt{x}$ *where* $x \in (0, \infty)$. *Prove that* f *is continuous at 1.*

Example 5.1.5 *Let* $f(x) = 3 - x^2$ *where* $x \in [-1, 2] \cup \{3\}$. *Prove that* f *is continuous at* $x = 3$

Example 5.1.6 *Prove that the function*

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0.

Theorem 5.1.7 *Let I be an open interval that contain a point a and $f : I \to \mathbb{R}$. Then*

$$f \text{ is continuous at } a \in I \text{ if and only if } \quad f(a) = \lim_{x \to a} f(x).$$

Example 5.1.8 *Let* $f(x) = x\cos\left(\dfrac{1}{x}\right)$ *where* $x \neq 0$. *If* f *is continuous at 0, what is* $f(0)$ *defined?*

Example 5.1.9 *Find* a *such that the function* $f(x) = \begin{cases} ax+1 & \text{if } x \geq 1 \\ 2x+3 & \text{if } x < 1 \end{cases}$ *is continuous at 1.*

Theorem 5.1.10 *Suppose that E is a nonempty subset of $\mathbb{R}$, $a \in E$, and $f : E \to \mathbb{R}$. Then the following statements are equivalent:*

1. *f is continuous at $a \in E$.*

2. *If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.*

Example 5.1.11 *Use Theorem 5.1.10 to find* $\lim_{n\to\infty} \sqrt{\dfrac{n}{n+1}}$.

Theorem 5.1.12 *Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. If f, g are continuous at a point $a \in E$, then so are*

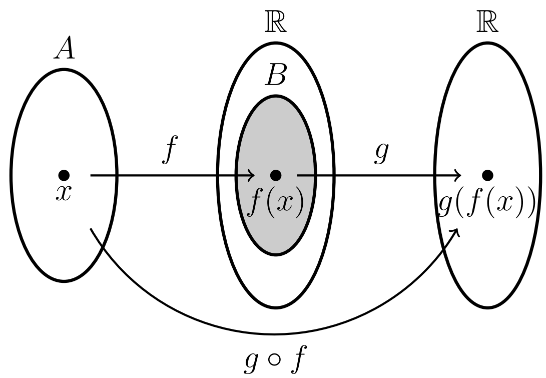
$$f + g, \quad fg \quad \textit{and} \quad \alpha f$$

Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$.

CONTINUITY OF COMPOSITION.

Definition 5.1.13 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $\{f(x) : x \in A\} \subseteq B$, then the composition of g with f is the function*

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$



Theorem 5.1.14 *Suppose that A and B are subsets of $\mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $\{f(x) : x \in A\} \subseteq B$. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then*

$$g \circ f \text{ is continuous at } a \in A$$

and moreover,

$$\lim_{x \to a} (g \circ f)(x) = g\left(\lim_{x \to a} f(x)\right).$$

Example 5.1.15 *Show that* $\lim_{x \to 1} \sqrt{2x-1}$ *exists and equals to 1.*

CONTINUITY ON A SET.

Definition 5.1.16 *Let* E *be a nonempty subset of* $\mathbb{R}$ *and* $f : E \to \mathbb{R}$.

> f *is said to be* ***continuous on E*** *if and only if* f *is continuous at every* $a \in E$.

Note that if f is continuous on E, then f is continuous on nonempty subset of E.

Example 5.1.17 *Show that* $f(x) = x^2$ *is continuous on* $\mathbb{R}$.

Theorem 5.1.18 (**Continuity of Linear function**) *Let m and c be constants and let*

$$f(x) = mx + c \text{ where } x \in \mathbb{R}.$$

Prove that f is continuous on $\mathbb{R}$

Example 5.1.19 *Show that $h(x) = (3x+1)^2$ is continuous on $\mathbb{R}$.*

Example 5.1.20 *Prove that*

$$f(x) = \begin{cases} 2x+4 & \text{if } x > -1 \\ 3x+5 & \text{if } x \le -1 \end{cases}$$

is continuous on $\mathbb{R}$.

Example 5.1.21 *Find* a *such that the function* $f(x) = \begin{cases} ax+1 & \text{if } x \ge 2 \\ x+a & \text{if } x < 2 \end{cases}$ *is continuous on* $\mathbb{R}$.

Exercises 5.1

1. Use definition to prove that f is continuous at a.

   1.1 $f(x) = x^2 + 1$ and $a = 1$.

   1.2 $f(x) = x^3$ and $a = -1$.

   1.3 $f(x) = \dfrac{1}{x}$ and $a = 1$.

   1.4 $f(x) = \dfrac{x}{x^2+1}$ and $a = 2$.

2. Determine whether the following functions are continuous at a.

   2.1 $f(x) = \begin{cases} 1-2x & \text{if } x \geq 1 \\ 2-3x & \text{if } x < 1 \end{cases}$ and $a = 1$

   2.2 $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 0 \\ \sqrt{1-x} & \text{if } x < 0 \end{cases}$ and $a = 0$

3. Use definition to prove that f is continuous at E.

   3.1 $f(x) = x^3$ and $E = \mathbb{R}$.

   3.2 $f(x) = \sqrt{1-x}$ and $E = (-\infty, 1)$.

   3.3 $f(x) = \dfrac{1}{x^2+1}$ and $E = \mathbb{R}$.

4. Use limit theorem to show that the following function are continuous on $[0, 1]$.

   4.1 $f(x) = 3x^2 + 1$

   4.2 $f(x) = \dfrac{1-x}{1+x}$

   4.3 $f(x) = \sqrt{2-x}$

   4.4 $f(x) = \dfrac{1}{x^2+x-6}$

5. Find a and b such that the function $f(x) = \begin{cases} ax+3 & \text{if } x \leq 1 \\ x+b & \text{if } 1 < x \leq 2 \\ 2ax-2 & \text{if } x > 2 \end{cases}$ is continuous on $\mathbb{R}$.

6. If $f : [a, b] \to \mathbb{R}$ is continuous, prove that $\sup\limits_{x\in[a,b]} |f(x)|$ is finite.

7. Show that there exist nowhere continuous functions f and g whose sum $f + g$ is continuous on $\mathbb{R}$. Show that the same is ture for product of functions.

8. Let
$$f(x) = \begin{cases} \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$
is continuous on $(-\infty, 0)$ and $(0, \infty)$, discontinuous at 0, and neither $f(0^+)$ nor $f(0^-)$ exists.

   8.1 Prove that f is continuous on $(-\infty, 0)$ and $(0, \infty)$ discontinuous at 0.

   8.2 Suppose that $g : [0, \frac{2}{\pi}] \to \mathbb{R}$ is continuous on $(0, \frac{2}{\pi})$ and that there is a positive constant $C > 0$ such that
$$|g(x)| \leq C\sqrt{x} \text{ for all } x \in (0, \tfrac{2}{\pi}),$$
   Prove that $f(x)g(x)$ is continuous on $[0, \frac{2}{\pi}]$.

9. Suppose that $a \in \mathbb{R}$, that I is an open interval containing a, that, $f, g : I \to \mathbb{R}$, and that f is continuous at a.

   9.1 Prove that g is continuous at a if and only if $f + g$ is continuous at a.

   9.2 Make and prove an analogous atstement for the product fg. Show by example that hypothesis about f added cannot be dropped.

10. Let $f : A \to \mathbb{R}$ be a continuous function. Suppose that $E \subseteq A$ and is open. Determine whether $\{f(x) : x \in E\}$ is open.

11. Let $f(x) = x^n$ where $n \in \mathbb{N}$. Prove that f is continuous on $\mathbb{R}$

12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

   12.1 Show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

   12.2 Prove that $f(qx) = qf(x)$ for all $x \in \mathbb{R}$ and $q \in \mathbb{Q}$.

   12.3 Prove that f is continuous at 0 if and only if f is continuous on $\mathbb{R}$.

   12.4 Prove that f is continuous at 0, then there is an $m \in \mathbb{R}$ such that $f(x) = mx$ for all $x \in \mathbb{R}$.

13. Assume that $\lim_{n\to 0} \dfrac{\ln(x+1)}{x} = 1$ and $f(x) = e^x$ is continuous on $\mathbb{R}$. Show that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

5.2 Intermediate Value Theorem

Definition 5.2.1 *Let E be a nonempty subsets of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be* ***bounded on E*** *if and only if there is an $M > 0$ such that*

$$|f(x)| \leq M \quad \text{for all } x \in E$$

Example 5.2.2 *Show that $f(x) = \dfrac{1}{x^2+1}$ is bounded on $\mathbb{R}$.*

Definition 5.2.3 *Let I be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous on I. Define*

$$\sup_{x \in I} f(x) := \sup\{f(x) : x \in I\} \quad \text{and} \quad \inf_{x \in I} f(x) := \inf\{f(x) : x \in I\}.$$

Example 5.2.4 *Let $f(x) = x^2$. Find a supremum and infimum of f on I.*

1. $I = [0, 1)$
2. $I = (-1, 1)$
3. $I = (-1, \infty)$

Theorem 5.2.5 (**Extreme Value Theorem (EVT)**) *If I is a closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover, if*

$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x),$$

then there exist point $x_m, x_M \in I$ such that

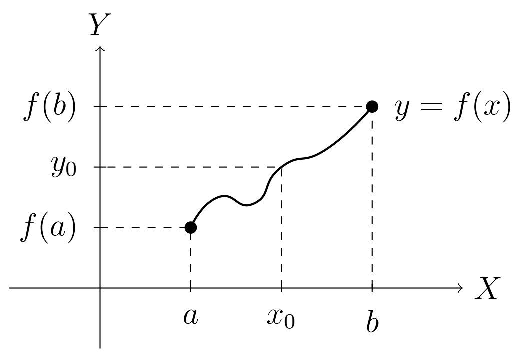
$$f(x_M) = M \quad \text{and} \quad f(x_m) = m.$$

Lemma 5.2.6 (**Sign-Preserving Property**) *Let* $f : I \to \mathbb{R}$ *where* I *is open. If* f *is continuous at a point* $x_0 \in I$ *and* $f(x_0) > 0$*, then there are positive numbers* ε *and* δ *such that*

$$|x - x_0| < \delta \quad \textit{implies} \quad f(x) > \varepsilon.$$

Theorem 5.2.7 (**Intermediate Value Theorem (IVT)**) *Let $f : [a, b] \to \mathbb{R}$ be continuous. If y_0 lies between $f(a)$ and $f(b)$, then*

$$\textit{there is an } x_0 \in (a, b) \textit{ such that } f(x_0) = y_0.$$



Corollary 5.2.8 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous.*

1. *If* $f(a) > 0$ *and* $f(b) < 0$*, then there is an* $c \in (a, b)$ *such that* $f(c) = 0$*.*

2. *If* $f(a) < 0$ *and* $f(b) > 0$*, then there is an* $c \in (a, b)$ *such that* $f(c) = 0$*.*

Example 5.2.9 *Show that there is a real number such that* $x^2 = x + 1$*.*

Example 5.2.10 *Show that there is a real number* x *such that* $x^3 - x - 3 = 0$*.*

Example 5.2.11 *Prove that*

$$\ln x = 3 - 2x$$

has at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ *of length 0.01 that contain a root.*

Exercises 5.2

For these exercise, assume that $\sin x$, $\cos x$ and e^x are continuous on $\mathbb{R}$ and $\ln x$ is continuous on $\mathbb{R}^+$.

1. For each of the following, prove that there is at least one $x \in \mathbb{R}$ that satisfies the given equation.

   1.1 $x^3 + x = 3$
   1.2 $x^3 + 2 = 2x$
   1.3 $x^4 + x^3 - 2 = 0$
   1.4 $x^5 + x + 1 = 0$
   1.5 $2^x = 2 - x$
   1.6 $e^x = x^2$
   1.7 $x \ln x = 1$
   1.8 $\sin x = e^x$
   1.9 $\cos x = x^2$
   1.10 $e^x = \cos x + 1$

2. Prove that the follwing equations have at least one real root and find the approximate root to be the midpont of an interval $[a, b]$ of length ℓ that contain a root.

   2.1 $x^3 + x = 1$ and $\ell = 0.001$
   2.2 $2^x = x^3$ and $\ell = 0.01$
   2.3 $\ln x + x = 2$ and $\ell = 0.001$
   2.4 $\cos x = x$ and $\ell = 0.01$
   2.5 $\sin x + x = 1$ and $\ell = 0.001$
   2.6 $xe^x = \cos x$ and $\ell = 0.01$

3. Suppose that f is a real-value function of a real variable. If f is continuous at a with $f(a) < M$ for some $M \in \mathbb{R}$, prove that there is an open interval I containing a such that
$$f(x) < M \text{ for all } x \in I.$$

4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and
$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty,$$
prove that f has a minimum on $\mathbb{R}$; i.e., there is an $x_m \in \mathbb{R}$ such that
$$f(x_m) = \inf_{x\in\mathbb{R}} f(x) < \infty.$$

5.3 Uniform continuity

Definition 5.3.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be* ***uniformly continuous on E*** *if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|x - a| < \delta \quad \textit{and} \quad x, a \in E \quad \textit{imply} \quad |f(x) - f(a)| < \varepsilon.$$

Example 5.3.2 *Prove that $f(x) = x$ is uniformly continuous on $(0, 1)$.*

Example 5.3.3 *Prove that $f(x) = x^2$ is uniformly continuous on $(0, 1)$.*

Theorem 5.3.4 (**Uniform of continuity of Linear function**) *A Linear function is uniformly continuous on* $\mathbb{R}$.

Example 5.3.5 *Prove that* $f(x) = x^2$ *is not uniformly continuous on* $\mathbb{R}$.

Theorem 5.3.6 *Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.*

Theorem 5.3.7 *Suppose that* $E \subseteq \mathbb{R}$ *and* $f : E \to \mathbb{R}$ *is uniformly continuous. If* $x_n \in E$ *is Cauchy, then* $f(x_n)$ *is Cauchy.*

Exercises 5.3

1. Use Definition to prove that each of the following functions is uniformly continuous on $(0, 1)$.

   1.1 $f(x) = x^3$

   1.2 $f(x) = x^2 - x$

   1.3 $f(x) = \dfrac{1}{x+1}$

2. Prove that each of the following functions is uniformly continuous on $(0, 1)$.

   2.1 $f(x) = (x+1)^2$

   2.2 $f(x) = \dfrac{x^3 - 1}{x - 1}$

   2.3 $f(x) = x \sin(\frac{1}{x})$

   2.4 $f(x)$ is any polynomial

   2.5 $f(x) = \dfrac{\sin x}{x}$

   2.6 $f(x) = x^2 \ln x$

3. Prove that $f(x) = \dfrac{1}{x^2+1}$ is uniformly continuous on $\mathbb{R}$.

4. Find all real α such that $x^\alpha \sin(\frac{1}{x})$ is uniformly continuous on the open interval $(0, 1)$.

5. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and there is an $L \in \mathbb{R}$ such that $f(x) \to L$ as $x \to \infty$. Prove that f is uniformly continuous on $[0, \infty)$.

6. Let I be a bounded interval. Prove that if $f : I \to \mathbb{R}$ is is uniformly continuous on I, then f is bounded on I.

7. Prove that (6) may be false if I is unbounded or if f is merely continuous.

8. Suppose that $\alpha \in \mathbb{R}$, E is nonempty subset of $\mathbb{R}$, and $f, g : E \to \mathbb{R}$ are uniformly continuous on E.

   8.1 Prove that $f + g$ and αf are uniformly continuous on E.

   8.2 Suppose that f, g are bounded on E. Prove that fg is uniformly continuous on E.

   8.3 Show that there exist functions f, g uniformly continuous on $\mathbb{R}$ such that fg is not uniformly continuous on $\mathbb{R}$.

9. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Chapter 6

Differentiability on $\mathbb{R}$

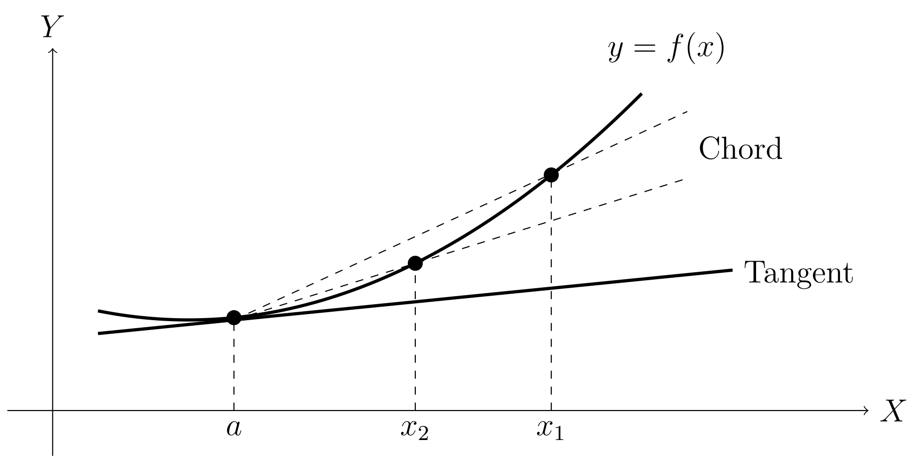
6.1 The Derivative

Definition 6.1.1 *A real function f is siad to be* ***differentiable*** *at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and*

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case $f'(a)$ is called the ***derivative*** *of f at a.*

You may recall that the graph of $y = f(x)$ has a **tangent line** at the point $(a, f(a))$ if and only if f has a derivative at a, in which case the slope of that tangent line is $f'(a)$. Suppose that f is differentiable at a. A **secant line** of the graph $y = f(x)$ is a line passing through at least two points on the graph, an a **chord** is a line segment that runs from one point on the graph to another.



Let $x = a + h$ and observe that the slope of the chord (chord function : $F(x)$) passing through the points $(x, f(x))$ and $(a, f(a))$ is given by

$$F(x) := \frac{f(x) - f(a)}{x - a}, \quad x \neq a.$$

Now, since $x = a + h$, $f'(a)$ becomes

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 6.1.2 *Let* $f(x) = x^2$ *where* $x \in \mathbb{R}$. *Find* $f'(1)$

Example 6.1.3 *Show that the function*

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at the origin.

Example 6.1.4 *Show that the function*

$$f(x) = \begin{cases} x\cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not differentiable at the origin.

Theorem 6.1.5 *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at a if and only if there is a function T of the form $T(x) := mx$ such that*

$$\lim_{h\to 0} \left| \frac{f(a+h) - f(a) - T(h)}{h} \right| = 0.$$

Theorem 6.1.6 *If f is differentiable at a, then f is continuous at a.*

Example 6.1.7 *Show that $f(x) = |x|$ is continuous at 0 but not differentiable there.*

DIFFERENTIABLE ON INTERVAL.

Definition 6.1.8 *Let I be an interval and $f : I \to \mathbb{R}$ be a function. f is said to be* ***differentiable on I*** *if and only if f is differentiable at a for every $a \in I$*

Example 6.1.9 *Show that the function $f(x) = x^2$ is differentiable on $\mathbb{R}$.*

Theorem 6.1.10 *Let $n \in \mathbb{N}$. If $f(x) = x^n$, then f is differentiable on $\mathbb{R}$ and*

$$f'(x) = nx^{n-1}.$$

Theorem 6.1.11 *Every constant function is differentiable on* $\mathbb{R}$ *and its value equals to zero.*

Example 6.1.12 *Show that* $f(x) = \sqrt{x}$ *is differentiable on* $(0, \infty)$ *and* $f'(x)$.

Example 6.1.13 *Show that* $f(x) = |x|$ *is differentiable on* $[0, 1]$ *and* $[-1, 0]$ *but not on* $[-1, 1]$.

Exercises 6.1

1. For each of the following real functions, use definition directly to prove that $f'(a)$ exists.

   1.1 $f(x) = x^3, \quad a \in \mathbb{R}$

   1.2 $f(x) = \dfrac{1}{x}, \quad a \neq 0$

   1.3 $f(x) = x^2 + x + 2, \quad a \in \mathbb{R}$

   1.4 $f(x) = \dfrac{1}{\sqrt{x}}, \quad a > 0$

2. Prove that $f(x) = x|x|$ is differentiable on $\mathbb{R}$.

3. Let I be an open interval that contains 0 and $f : I \to \mathbb{R}$. If there exists an $\alpha > 1$ such that

$$|f(x)| \leq |x|^\alpha \text{ for all } x \in I,$$

   prove that f is differentiable at 0. What happens when $\alpha = 1$?

4. Suppose that $f : (0, \infty) \to \mathbb{R}$ satisfies $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$ for all $x, y \in (0, \infty)$ and $f(1) = 0$.

   4.1 Prove that f is continuous on $(0, \infty)$ if and only if f is continuous at 1.

   4.2 Prove that f is differentiable on $(0, \infty)$ if and only if f is differentiable at 1.

   4.3 Prove that if f is differentiable at 1, then $f'(x) = \dfrac{f'(1)}{x}$ for all $x \in (0, \infty)$.

5. Suppose that $f_\alpha(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that $f_\alpha(x)$ is continuous at $x = 0$ when $\alpha > 0$ and differentiable at $x = 0$ when $\alpha > 1$. Graph these functions for $\alpha = 1$ and $\alpha = 2$ and give a geometric interpretation of your results.

6. Prove that if $f(x) = x^\alpha$ where $\alpha = \frac{1}{n}$ for somw $n \in \mathbb{N}$, then $y = f(x)$ is differentiable on $f'(x) = \alpha x^{\alpha - 1}$ for every $x \in (0, \infty)$.

7. Given $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. Show that

   7.1 $(\sin x)' = \cos x$

   7.2 $(\cos x)' = -\sin x$

8. f is a constant function on I if and only if $f'(x) = 0$ for every $x \in I$.

6.2 Differentiability theorem

Theorem 6.2.1 (**Additive Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $f+g$ is differentiable at a. In fact,*

$$(f+g)'(a) = f'(a) + g'(a).$$

Theorem 6.2.2 (**Scalar Multiplicative Rule**) *Let f be a real function and $\alpha \in \mathbb{R}$. If f is differentiable at a, then αf is differentiable at a. In fact,*

$$(\alpha f)'(a) = \alpha f'(a).$$

Theorem 6.2.3 (**Product Rule**) *Let f and g be real functions. If f and g are differentiable at a, then fg is differentiable at a. In fact,*

$$(fg)'(a) = g(a)f'(a) + f(a)g'(a).$$

Theorem 6.2.4 (**Quotient Rule**) *Let f and g be real functions. If f and g are differentiable at a, then $\frac{f}{g}$ is differentiable at a when $g(a) \neq 0$. In fact,*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}.$$

Example 6.2.5 *Let f and g be differentiable at 1 with $f(1) = 1, g(1) = 2$ and $f'(1) = 3, g'(1) = 4$. Evaluate the following derivatives.*

1. $(f + g)'(1)$
2. $(2f)'(1)$
3. $(fg)'(1)$
4. $\left(\frac{f}{g}\right)'(1)$

Theorem 6.2.6 (Chain Rule) *Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Example 6.2.7 *Let f and g be differentiable on $\mathbb{R}$ with $f(0) = 1, g(0) = -1$ and $f'(0) = 2$, $g'(0) = -2$,$f'(-1) = 3$, $g'(1) = 4$. Evaluate each of the following derivatives.*

1. $(f \circ g)'(0)$
2. $(g \circ f)'(0)$

Example 6.2.8 *Let $f(x) = \sqrt{x^2 + 1}$. Use the Chain Rule to show that $f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$.*

Exercises 6.2

1. For each of the following functions, find all x for which $f'(x)$ exists and find a formula for f'.

   1.1 $f(x) = \dfrac{x^3 - 2x^2 + 3x}{\sqrt{x}}$

   1.2 $f(x) = \dfrac{1}{x^2 + x - 1}$

   1.3 $f(x) = x|x|$

   1.4 $f(x) = |x^3 + 2x^2 - x - 2|$

2. Let f and g be differentiable at 2 and 3 with $f'(2) = a$, $f'(3) = b$, $g'(2) = c$ and $g'(3) = d$, If $f(2) = 1$, $f(3) = 2$,$g(2) = 3$ and $g(3) = 4$. Evaluate each of the following derivatives.

   2.1 $(fg)'(2)$

   2.2 $\left(\frac{f}{g}\right)'(3)$

   2.3 $(g \circ f)'(3)$

   2.4 $(f \circ g)'(2)$

3. If f, g and h is differentiable at a, prove that fgh is differentiable at a and

$$(fgh)'(a) = f'(a)g(a)h(a) + f(a)g'(a)h(a) + f(a)g(a)h'(a).$$

4. Let $f(x) = (x-1)(x-2)(x-3)\cdots(x-2565)$. Find $f'(2565)$

5. Prove that if $f(x) = x^{\frac{m}{n}}$ for some $n, m \in \mathbb{N}$, then $y = f(x)$ is differentiable and satisfies $ny^{n-1}y' = mx^{m-1}$ for every $x \in (0, \infty)$.

6. (**Power Rule**) Prove that $f(x) = x^q$ for some $q \in \mathbb{Q}$, then f is differentiable and $f'(x) = qx^{q-1}$ for every $x \in (0, \infty)$.

7. **(Reciprocal Rule)** Suppose that f is differentiable at a and $f(a) \neq 0$.

   7.1 Show that for h sufficiently small, $f(a+h) \neq 0$.

   7.2 Use Definition 6.1.1 directly, prove that $\dfrac{1}{f(x)}$ is differentiable at $x = a$ and

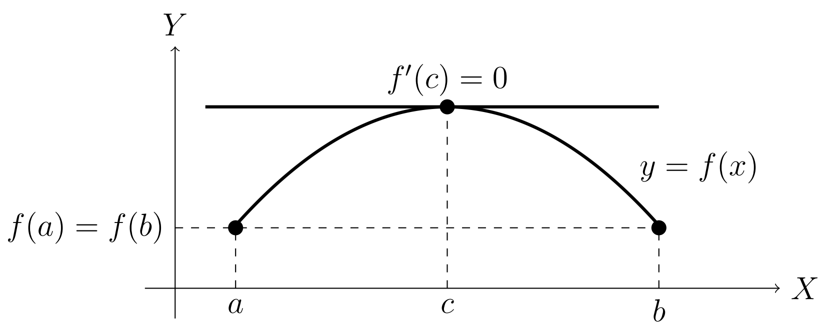
$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{f^2(a)}.$$

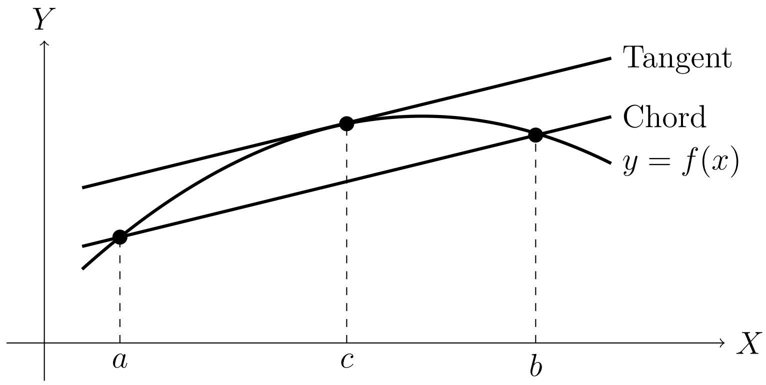
8. Suppose hat $n \in \mathbb{N}$ and f, g are real functions of a real variable whose nth derivatives $f^{(n)}, g^{(n)}$ exist at a point a. Prove Leibniz's generalization of the Product Rule:

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a).$$

6.3 Mean Value Theorem

Lemma 6.3.1 (**Rolle's Theorem**) *Suppose that* $a, b \in \mathbb{R}$ *with* $a \neq b$*. If* f *is continuous on* $[a, b]$*, differentiable on* (a, b)*, and if* $f(a) = f(b)$*, then* $f'(c) = 0$ *for some* $c \in (a, b)$*.*





Theorem 6.3.2 (**Mean Value Theorem (MVT)**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is an $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Example 6.3.3 *Prove that*

$$\sin x \leq x \quad \textit{for all } x > 0.$$

Example 6.3.4 *Prove that*

$$1 + x \leq e^x \quad \textit{for all } x > 0.$$

Example 6.3.5 (**Bernoulli's Inequality**) *Let* $0 < \alpha \leq 1$ *and* $\delta \geq -1$. *Prove that*

$$(1+\delta)^{\alpha} \leq 1 + \alpha\delta.$$

Theorem 6.3.6 (**Generalized Mean Value Theorem**) *Suppose that $a, b \in \mathbb{R}$ with $a \neq b$. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is an $c \in (a, b)$ such that*

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)].$$

Theorem 6.3.7 (L'Hôspital's Rule) *Let a be an extended real number and I be an open interval that either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I\backslash\{a\}$, and $g(x) \neq 0 \neq g'(x)$ for all $x \in I\backslash\{a\}$. Suppose further that*

$$A := \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

is either 0 or ∞. If

$$B := \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Given $(\ln x)' = \dfrac{1}{x}$ for $x > 0$ and $(e^x)' = e^x$ for all $x \in \mathbb{R}$.

Example 6.3.8 *Use L'Hôspital's Rule to prove that* $\displaystyle\lim_{x\to 0} \frac{x}{e^x - 1} = 1.$

Example 6.3.9 *Use L'Hôspital's Rule to find* $\displaystyle\lim_{x\to 0^+} x \ln x.$

Example 6.3.10 *Use L'Hôspital's Rule to find* $\displaystyle L = \lim_{x\to 1^-} (\ln x)^{1-x}.$

Exercises 6.3

1. Use the Mean Value Theorem to prove that each of the following inequalities.

   1.1 $\sqrt{2x+1} < 1+x$ for all $x > 0$

   1.2 $\ln x \le x-1$ for all $x > 1$

   1.3 $7(x-1) < e^x$ for all $x > 2$

   1.4 $\cos x - 1 \le x$ for all $x > 0$

   1.5 $\ln x + 1 \le \dfrac{x^2+1}{2}$ for all $x > 1$

   1.6 $\dfrac{x-1}{x} \le \ln x$ for all $x > 1$

   1.7 $\sqrt{x} \ge x$ for all $x \in [0,1]$

   1.8 $\sqrt{x} \le x$ for all $x > 1$

   1.9 $\sin^2 x \le 2|x|$ for all $x \in \mathbb{R}$

   1.10 $\ln x \le \sqrt{x}$ for all $x > 1$

2. (**Bernoulli's Inequality**) Let $\alpha \ge 1$ and $\delta \ge -1$. Prove that

$$(1+\delta)^\alpha \le 1 + \alpha\delta.$$

3. Use L'H***ô***spital's Rule to evaluate the following limits.

   3.1 $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

   3.2 $\displaystyle\lim_{x\to 0^+} \frac{\cos x - e^x}{\ln(1+x^2)}$

   3.3 $\displaystyle\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{1}{x^2}}$

   3.4 $\displaystyle\lim_{x\to 0^+} x^x$

   3.5 $\displaystyle\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)}$

   3.6 $\displaystyle\lim_{x\to \infty} x\left(\arctan x - \frac{\pi}{2}\right)$

   3.7 $\displaystyle\lim_{x\to 0^-} \left(1+e^{-x}\right)^x$

   3.8 $\displaystyle\lim_{x\to 0} (1+x)^{\frac{1}{x}}$

   3.9 $\displaystyle\lim_{x\to \infty} x(e^{\frac{1}{x}}-1)$

4. Show that the derivative of

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   exists and continuous on $\mathbb{R}$ with $f'(0) = 0$.

5. Suppose that f is differentiable on $\mathbb{R}$.

   5.1 If $f'(x) = 0$ for all $x \in \mathbb{R}$, prove that $f(x) = f(0)$ for all $x \in \mathbb{R}$

   5.2 If $f(0) = 1$ and $|f'(x)| \le 1$ for all $x \in \mathbb{R}$, prove that $|f(x)| \le |x| + 1$ for all $x \in \mathbb{R}$

   5.3 If $'(x) \ge 0$ for all $x \in \mathbb{R}$, prove that $a < b$ imply that $f(a) < f(b)$

6. Let f be differentiable on a nonempty, open interval (a, b) with f' bounded on (a, b). Prove that f is uniformly continuous on (a, b).

7. Let f be differentiable on (a, b), continuous on $[a, b]$, with $f(a) = f(b) = 0$. Prove that if $f'(c) > 0$ for some $c \in (a, b)$, then there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) > 0 > f'(x_2)$.

8. Let f be twice differentiable on (a, b) and let there be points $x_1 < x_2 < x_3$ in (a, b) such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there is a point $c \in (a, b)$ such that $f''(c) > 0$.

9. Let f be differentiable on $(0, \infty)$. If $L = \lim_{x\to\infty} f'(x)$ and $\lim_{n\to\infty} f(n)$ both exist and are finite, prove that $L = 0$.

10. Prove L'Hôspital's Rule for the case $B = \pm\infty$ by first proving that

$$\frac{g(x)}{f(x)} \to 0 \text{ when } \frac{f(x)}{g(x)} \to \pm\infty, \text{ as } x \to a.$$

11. Prove that the sequence $\left(1 + \dfrac{1}{n}\right)^n$ is increasing, as $n \to \infty$, and its limit e satisfies $2 < e \leq 3$ and $\ln e = 1$.

6.4 Monotone function

Definition 6.4.1 *Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.*

1. *f is said to be* ***increasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \leq f(x_2).$$

   f is said to be ***strictly increasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) < f(x_2).$$

2. *f is said to be* ***decreasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) \geq f(x_2).$$

   f is said to be ***strictly decreasing*** *on E if and only if*

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \text{ imply } f(x_1) > f(x_2).$$

3. *f is said to be* ***monotone*** *on E if and only if f is either decreasing or increasing on E.*
   f is said to be ***strictly monotone*** *on E if and only if f is either strictly decreasing or strictly increasing on E.*

Example 6.4.2 *Show that $f(x) = x^2$ is strictly monotone on $[0, 1]$ and on $[-1, 0]$ but not monotone on $[-1, 1]$.*

Theorem 6.4.3 *Let $f : I \to \mathbb{R}$ and $(a, b) \subseteq I$. Then*

1. *f is increasing on (a, b) if $f'(x) > 0$ for all $x \in (a, b)$*

2. *f is decreasing on (a, b) if $f'(x) < 0$ for all $x \in (a, b)$*

3. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Example 6.4.4 *Find each intervals of $f(x) = x^2 - 4x + 3$ that increasing and decreasing.*

Theorem 6.4.5 *If f is 1-1 and continuous on an interval I, then f is strictly monotone on I and f^{-1} is continuous and strictly monotone on $f(I) := \{f(x) : x \in I\}$.*

Theorem 6.4.6 (**Inverse Function Theorem (IFT)**) *Let f be 1-1 and continuous on an open interval I. If $a \in f(I)$ and if $f'(f^{-1}(a))$ exists and is nonzero, then f^{-1} is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Example 6.4.7 *Use the IVT to find derivative of $f(x) = \arcsin x$*

Example 6.4.8 *Let* $f(x) = x + e^x$ *where* $x \in \mathbb{R}$.

1. *Show that* f *is 1-1 on* $x \in \mathbb{R}$.

2. *Use the result from 1 and the IFT to explain that* f^{-1} *differentiable on* $\mathbb{R}$.

3. *Compute* $(f^{-1})'(2 + \ln 2)$.

Exercises 6.4

1. Find each intervals of the following functions that increasing and decreasing.

   1.1 $f(x) = 2x - x^2$

   1.2 $f(x) = x^3 - x^2 - x + 3$

   1.3 $f(x) = (x-1)^3(x-2)^4$

   1.4 $g(x) = xe^x$

   1.5 $g(x) = e^x - x$

   1.6 $g(x) = x^2e^{x^2}$

2. Find all $a \in \mathbb{R}$ such that $x^3 + ax^2 + 3x + 15$ is strictly increasing near $x = 1$.

3. Find all $a \in \mathbb{R}$ such that $ax^2 + 3x + 5$ is strictly increasing on the interval $(1, 2)$.

4. Find where $f(x) = 2|x-1| + 5\sqrt{x^2+9}$ is strictly increasing and where $f(x)$ is strictly decreasing.

5. Let f and g be 1-1 and continuous on $\mathbb{R}$. If $f(0) = 2$, $g(1) = 2$, $f'(0) = \pi$, and $g'(1) = e$, compute the following derivatives.

   5.1 $(f^{-1})'(2)$

   5.2 $(g^{-1})'(2)$

   5.3 $(f^{-1} \cdot g^{-1})'(2)$

6. Let $f(x) = x^2e^{x^2}$, $x \in \mathbb{R}$.

   6.1 Show that f^{-1} exists and its differentiable on $(0, \infty)$.

   6.2 Compute $(f^{-1})'(e)$

7. Let $f(x) = x + e^{2x}$ where $x \in \mathbb{R}$.

   7.1 Show that f is 1-1 on $x \in \mathbb{R}$.

   7.2 Use the result from 7.1 and the IFT to explain that f differentiable on $\mathbb{R}$.

   7.3 Compute $(f^{-1})'(4 + \ln 2)$.

8. Use the Inverse Function Theorem, prove that

   8.1 $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$ where $x \in (-1, 1)$

   8.2 $(\arctan x)' = \dfrac{1}{1+x^2}$ where $x \in (-\infty, \infty)$

8.3 $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$ where $x \in (0, \infty)$

9. Use the IFT to find derivative of invrese function $f(x) = e^x - e^{-x}$ where $x \in \mathbb{R}$.

10. Suppose that f' exists and continuous on a nonempty, open interval (a, b) with $f'(x) \neq 0$ for all $x \in (a, b)$.

    10.1 Prove that f is 1-1 on (a, b) and takes (a, b) onto some open interval (c, d)

    10.2 Show that $(f^{-1})'$ exists and continuous on (c, d)

    10.3 Use the function $f(x) = x^3$, show that 7.2 is false if the assumption $f'(x) \neq 0$ fails to hold for some $x \in (c, d)$

11. Let $[a, b]$ be a closed, bounded interval. Find all functions f that satisfy the following conditions for some fixed $\alpha > 0$: f is continuous and 1-1 on $[a, b]$,

$$f'(x) \neq 0 \text{ and } f'(x) = \alpha(f^{-1})'(f(x)) \text{ for all } x \in (a, b).$$

12. Let f be differentiable at every point in a closed, bounded interval $[a, b]$. Prove that if f' is increasing on (a, b), then f' is continuous on (a, b).

13. Suppose that f is increasing on $[a, b]$. Prove that

    13.1 if $x_0 \in [a, b)$, then $f(x_0^+)$ exists and $f(x_0) \leq f(x_0^+)$,

    13.2 if $x_0 \in (a, b]$, then $f(x_0^-)$ exists and $f(x_0^-) \leq f(x_0)$.

Chapter 7

Integrability on $\mathbb{R}$

7.1 Riemann integral

PARTITION.

Definition 7.1.1 *Let $a, b \in \mathbb{R}$ with $a < b$.*

1. *A **partition** of the interval $[a, b]$ is a set of points $P = \{x_0, x_1, , ..., x_n\}$ such that*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

2. *The **norm** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is the number*

$$\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|.$$

3. *A **refinement** of a partition $P = \{x_0, x_1, , ..., x_n\}$ is a partition Q of $[a, b]$ that satisfies $Q \supseteq P$. In this case we say that Q is **finer** than P or Q is a **refinement** of P.*

Example 7.1.2 *Give example of partition and refinement of the interval $[0, 1]$.*

Partitions	Norms of Partition
$P = \{0, 0.5, 1\}$	
$Q = \{0, 0.25, 0.5, 0.75, 1\}$	
$R = \{0, 0.2, 0.3, , 0.5, 0.6, 0.8, 1\}$	

Example 7.1.3 *Prove that for each $n \in \mathbb{N}$,*

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

is a partition of the interval $[0, 1]$ and find a norm of P_n.

Example 7.1.4 (Dyadic Partition) *Let $n \in \mathbb{N}$ and define*

$$P_n = \left\{ \frac{j}{2^n} : j = 0, 1, ..., 2^n \right\}.$$

1. *Prove that P_n is a partition of the interval $[0, 1]$.*

2. *Prove that P_m is finer than P_n when $m > n$.*

3. *Find a norm of P_n.*

UPPER AND LOWER RIEMANN SUM.

Definition 7.1.5 *Let $a, b \in \mathbb{R}$ with $a < b$, let $P = \{x_0, x_1, , ..., x_n\}$ be a partition of the interval $[a, b]$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded.*

1. *The **upper Riemann sum** of f over P is the number*

$$U(f, P) := \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1})$$

where

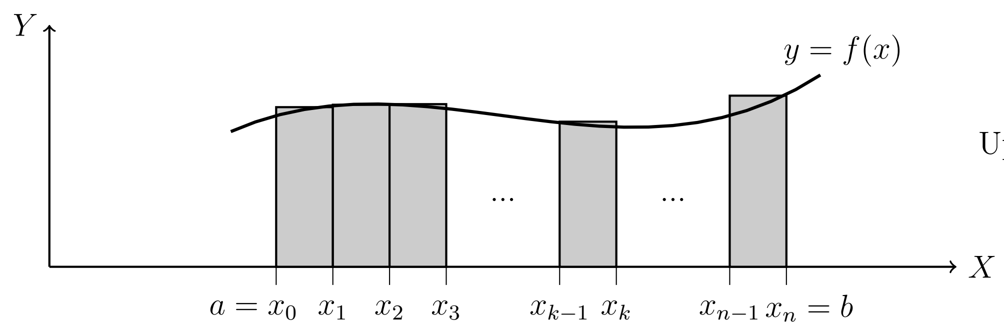
$$M_j(f) := \sup_{x \in [x_{j-1}, x_j]} f(x).$$

2. *The **lower Riemann sum** of f over P is the number*

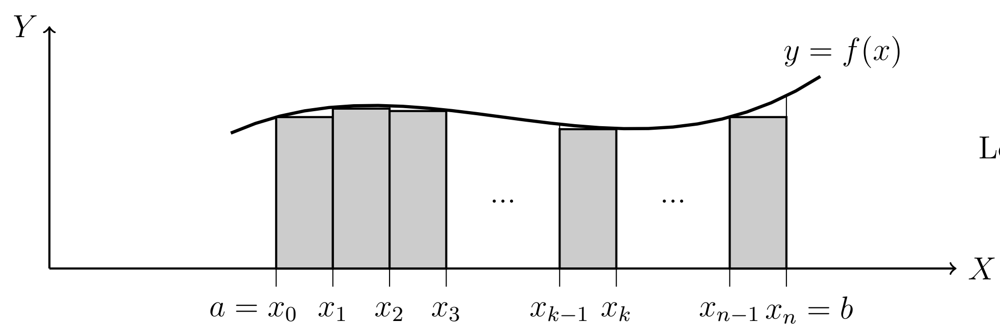
$$L(f, P) := \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1})$$

where

$$m_j(f) := \inf_{x \in [x_{j-1}, x_j]} f(x).$$



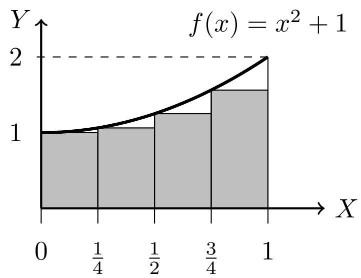
Upper Riemann Sum

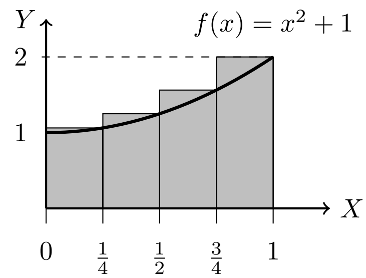


Lower Riemann Sum

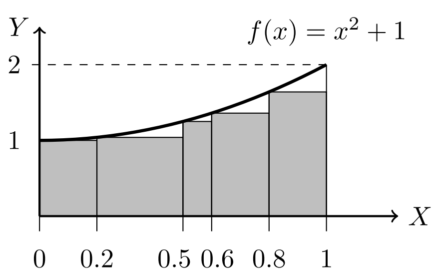
Example 7.1.6 *Let $f(x) = x^2 + 1$ where $x \in [0, 1]$. Find $L(f, P)$ and $U(f, P)$*

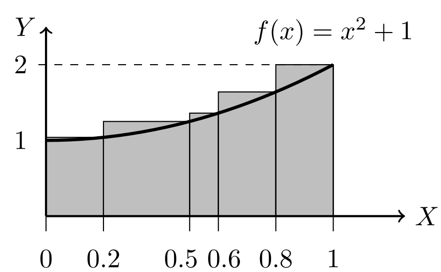
1. $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$





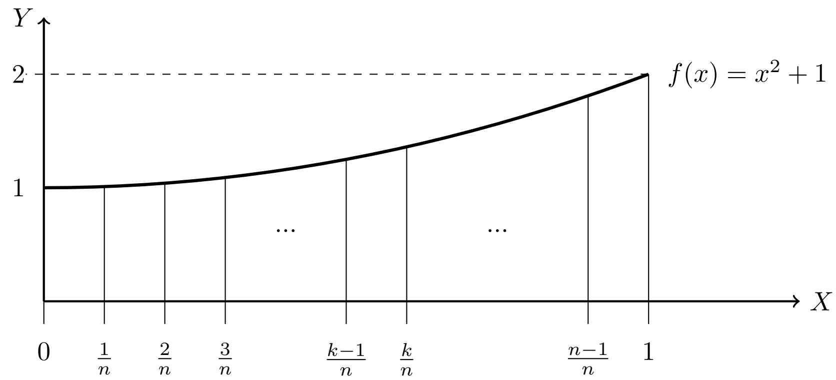
2. $P = \{0, 0.2, 0.5, 0.6, 0.8, 1\}$





Example 7.1.7 *Let* $f(x) = x^2 + 1$ *where* $x \in [0, 1]$. *Find* $L(P_n, f)$ *and* $U(P_n, f)$ *for* $n \in \mathbb{N}$ *if*

$$P_n = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}.$$



Theorem 7.1.8 *$L(f,P) \leq U(f,P)$ for all partition P and all bounded function f.*

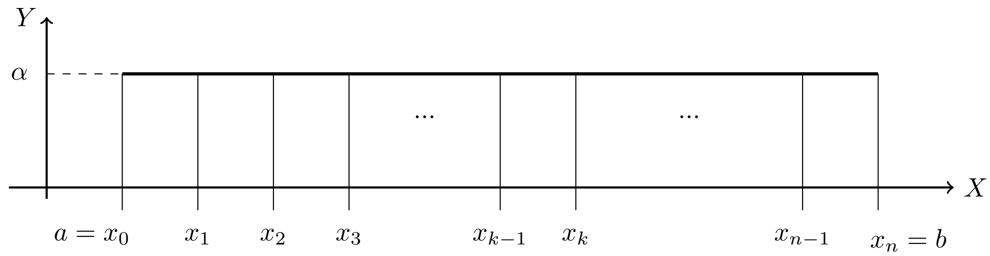
Theorem 7.1.9 (Sum Telescopes) *If $g : \mathbb{N} \to \mathbb{R}$, then*

$$\sum_{k=m}^{n} [g(k+1) - g(k)] = g(n+1) - g(m)$$

for all $n \geq m$ in $\mathbb{N}$.

Theorem 7.1.10 *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$U(f, P) = L(f, P) = \alpha(b - a)$$



Theorem 7.1.11 *If P is any partition of $[a,b]$ and Q is a refinement of P, then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Corollary 7.1.12 *If P and Q are any partitions of $[a,b]$, then*

$$L(f,P) \leq U(f,Q).$$

RIEMANN INTEGRABLE.

Definition 7.1.13 *Let $a, b \in \mathbb{R}$ with $a < b$.*

*A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** or **integrable** on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\varepsilon > 0$ there is a partition of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Theorem 7.1.14 *Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.*

Example 7.1.15 *Prove that the function*

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

is integrable on $[0, 1]$.

Example 7.1.16 (**Dirichlet function**) *Prove that the function*

$$f(x) = \begin{cases} 1 & \textit{if } x \in \mathbb{Q} \\ 0 & \textit{if } x \notin \mathbb{Q} \end{cases}$$

is NOT Riemann integrable on $[0, 1]$.

UPPER AND LOWER INTEGRABLE.

Definition 7.1.17 *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded.*

1. *The* ***upper integral*** *of f on $[a, b]$ is the number*

$$(U)\int_a^b f(x)\,dx := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

2. *The* ***lower integral*** *of f on $[a, b]$ is the number*

$$(L)\int_a^b f(x)\,dx := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

3. *If the upper and lower integrals of f on $[a, b]$ are equal, we define the* ***integral*** *of f on $[a, b]$ to be the common value*

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

Example 7.1.18 *Let $f(x) = \alpha$ where $x \in [a, b]$. Show that*

$$(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx = \alpha(b-a).$$

Example 7.1.19 *The Dirichlet function is defined*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

Find the upper integral and lower integral of the Dirichlet function on $[0, 1]$.

Theorem 7.1.20 *If* $f : [a,b] \to \mathbb{R}$ *is bounded, then its upper and lower integrals exist and are finite, and satisfy*

$$(L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx.$$

Theorem 7.1.21 *Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if*

$$(L) \int_a^b f(x)\, dx = (U) \int_a^b f(x)\, dx.$$

Theorem 7.1.22 *For a constant* α*,*

$$\int_a^b \alpha\, dx = \alpha(b-a).$$

Example 7.1.23 *Let* $f : [0,2] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

Show that f *is integrable and find* $\displaystyle\int_0^2 f(x)dx$*.*

Example 7.1.24 *Let* $f : [0, 1] \to \mathbb{R}$ *defined by*

$$f(x) = \begin{cases} 2 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Show that f *is integrable and find* $\displaystyle\int_0^1 f(x)dx$.

Exercises 7.1

1. For each of the following, compute $U(f,P)$, $L(f,P)$, and $\int_0^1 f(x)\,dx$, where

$$P = \left\{0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1\right\}.$$

Find out whether the lower sum or the upper sum is better approximation to the integral. Graph f and explain why this is so.

1.1 $f(x) = 1 - x^2$ 1.2 $f(x) = 2x^2 + 1$ 1.3 $f(x) = x^2 - x$

2. Let $P_n = \left\{\frac{j}{n} : n = 0, 1, ..., n\right\}$ for each $n \in \mathbb{N}$. Prove that a bounded function f is integrable on $[0,1]$ if

$$I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

in which case $\int_0^1 f(x)\,dx$ equals I_0.

3. For each of the following functions, use P_n in 2. to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)\,dx$.

3.1 $f(x) = x$

3.2 $f(x) = x^2$

3.3 $f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

4. Let $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that the function $f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } \text{ otherwise} \end{cases}$ is integrable on $[0,1]$. What is the value of $\int_0^1 f(x)\,dx$?

5. Suppose that f is continuous on an interval $[a,b]$. Show that $\int_a^c f(x)\,dx = 0$ for all $c \in [a,b]$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

6. Let f be bounded on a nondegenerate interval $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ such that

$$P \supseteq P_\varepsilon \quad \text{imples} \quad |U(f,P) - L(f,P)| < \varepsilon.$$

7.2 Riemann sums

Definition 7.2.1 *Let $f : [a, b] \to \mathbb{R}$.*

1. *A **Riemann sum** of f with respect to a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ is a sum of the form*

$$\sum_{j=1}^{n} f(t_j)\Delta x_j,$$

where the choice of $t_j \in [x_{j-1}, x_j]$ is arbitrary.

2. *The Riemann sums of f are **converge** to $I(f)$ as $\|P\| \to 0$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that*

$$P = \{x_0, x_1, ..., x_n\} \supseteq P_\varepsilon \quad \textit{implies} \quad \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \varepsilon$$

for all choice of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$. In this case we shall use the notation

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j.$$

Example 7.2.2 *Let* $f(x) = x^2$ *where* $x \in [0,1]$ *and*

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}$$

be a partition of $[0,1]$. *Show that if* $f(t_i)$ *is choosen by the right end point and left end point in each subinterval, then two* $I(f)$, *depend on two methods, are NOT different.*

Theorem 7.2.3 *Let $a, b \in \mathbb{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Then f is Riemann integrable on $[a, b]$ if and only if*

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j) \Delta x_j$$

exists, in which case

$$I(f) = \int_a^b f(x)\, dx.$$

Theorem 7.2.4 (**Linear Property**) *If f, g are integrable on $[a, b]$ and $\alpha \in \mathbb{R}$, then $f + g$ and αf are integrable on $[a, b]$. In fact,*

1. $\displaystyle\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
2. $\displaystyle\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx$

Theorem 7.2.5 *If f is integrable on $[a, b]$, then f is integrable on each subinterval $[c, d]$ of $[a, b]$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for all $c \in (a, b)$.

By Theorem 7.2.5, we obtain

$$\int_a^b f(x)\,dx = \int_a^a f(x)\,dx + \int_a^b f(x)\,dx$$

Thus,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Example 7.2.6 *Using the connection between integrals are area, evaluate* $\int_0^5 |x-2|\,dx$.

Example 7.2.7 *Using the connection between integrals are area, evaluate* $\int_0^2 \sqrt{4-x^2}\,dx$.

Theorem 7.2.8 (**Comparison Theorem**) *If f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

In particular, if $m \leq f(x) \leq M$ for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Theorem 7.2.9 *If f is Riemann integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Exercises 7.2

1. Using the connection between integrals are area, evaluate each of the following integrals.

1.1 $\displaystyle\int_0^1 |x-0.5|\,dx$

1.2 $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx, \quad a>0$

1.3 $\displaystyle\int_{-2}^2 (|x+1|+|x|)\,dx$

1.4 $\displaystyle\int_a^b (3x+1)\,dx, \quad a<b$

2. Prove that if f is integrable on $[0,1]$ and $\beta>0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{\frac{1}{n^\beta}} f(x)\,dx = 0 \qquad \text{for all } \alpha<\beta.$$

3. If f,g are integrable on $[a,b]$ and $\alpha\in\mathbb{R}$, prove that

$$\left|\int_a^b (f(x)+g(x))\,dx\right| \le \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx.$$

4. Suppose that $g_n\ge 0$ is a sequence of integrable function that satisfies $\displaystyle\lim_{n\to\infty}\int_a^b g_n(x)\,dx = 0$.
Show that if $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$, then $\displaystyle\lim_{n\to\infty}\int_a^b f(x)g_n(x)\,dx = 0$.

5. Prove that if f is integrable on $[0,1]$, then $\displaystyle\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx = 0$.

6. Prove that if f is integrable on $[0,1]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=0}^{n}\int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} f(x)\,dx.$$

7. Let f be continuous on a closed, nondegenerate interval $[a,b]$ and set $M=\displaystyle\sup_{x\in[a,b]}|f(x)|$.

7.1 Prove that if $M>0$ and $p>0$, then for every $\varepsilon>0$ there is a nondegenerate on interval $I\subset[a,b]$ such that

$$(M-\varepsilon)^p|I| \le \int_a^b |f(x)|^p\,dx \le M^p(b-a).$$

7.2 Prove that $\displaystyle\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} = M.$

7.3 Fundamental Theorem of Calculus

Define a set $C^1[a,b] = \{f : [a,b] \to \mathbb{R} : f$ is differentiable and f' are continuous $\}$ and $f'(x) = \dfrac{df}{dx}$.

Theorem 7.3.1 (**Fundamental Theorem of Calculus**) *Suppose that* $f : [a,b] \to \mathbb{R}$.

1. *If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F \in C^1[a,b]$ and*

$$\frac{d}{dx}\int_a^x f(t)\,dt := F'(x) = f(x)$$

   for each $x \in [a,b]$.

2. *If f is differentiable on $[a,b]$ and f' is integrable on $[a,b]$, then*

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

   for each $x \in [a,b]$.

Example 7.3.2 *Assume that f is differentiable on $(0,1)$ and integrable on $[0,1]$. Show that*

$$\int_0^1 xf'(x) + f(x)\,dx = f(1).$$

Theorem 7.3.3 *Let $\alpha \neq -1$. Then*

$$\int_a^b x^\alpha\,dx = f(b) - f(a) \quad \text{where } f(x) = \frac{x^{\alpha+1}}{\alpha+1}.$$

Example 7.3.4 *Find integral* $\int_0^1 x^2\,dx$.

Theorem 7.3.5 *Suppose that* $f, u : [a, b] \to \mathbb{R}$*. If* f *is continuous on* $[a, b]$ *and* $F(x) = \int_a^{u(x)} f(t)\, dt$*, and* $F \in C^1[a, b]$ *and*

$$F'(x) = \frac{d}{dx} \int_a^{u(x)} f(t)\, dt = f(u(x)) \cdot u'(x)$$

for each $x \in [a, b]$*.*

Example 7.3.6 *Let* $F(x) = \int_0^{\sin x} e^{t^2}\, dt$*. Find* $F(0)$ *and* $F'(0)$*.*

INTEGRATION BY PART.

Theorem 7.3.7 (**Integration by Part**) *Suppose that f, g are differentiable on $[a, b]$ with f', g' integrable on $[a, b]$, Then*

$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$

Example 7.3.8 *Use the Integration by Part to find integrals.*

1. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$

2. $\displaystyle\int_1^2 \ln x\,dx$

Example 7.3.9 *Let* $f(x) = \int_0^{x^3} e^{t^2} dt$. *Use integration by part to show that*

$$6\int_0^1 x^2 f(x)dx - 2\int_0^1 e^{x^2} dx = 1 - e.$$

CHANGE OF VARIABLES.

Theorem 7.3.10 (**Change of Variables**) *Let ϕ be continuously differentiable on a closed interval $[a, b]$. If f is continuous on $\phi([a, b])$, or if ϕ is strictly incresing on $[a, b]$ and f is integrable on $[\phi(a), \phi(b)]$, then*

$$\int_{\phi(a)}^{\phi(b)} f(t)\, dt = \int_a^b f(\phi(x))\phi'(x)\, dx.$$

Example 7.3.11 *Find* $\displaystyle\int_0^3 \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx$

Example 7.3.12 *Evaluate*

$$\int_{-1}^{1} x f(x^2)\, dx$$

for any f *is continuous on* $[0, 1]$.

Example 7.3.13 *Let* $f : [-a, a] \to \mathbb{R}$ *where* $a > 0$. *Suppose* $f(-x) = -f(x)$ *for all* $x \in [-a, a]$. *Show that*

$$\int_{-a}^{a} f(x)\, dx = 0.$$

Exercises 7.3

1. Compute each of the following integrals.

1.1 $\displaystyle\int_{-3}^{3} \lvert x^2+x-2\rvert\,dx$	1.4 $\displaystyle\int_{1}^{e} x\ln x\,dx$
1.2 $\displaystyle\int_{1}^{4} \frac{\sqrt{x}-1}{\sqrt{x}}\,dx$	1.5 $\displaystyle\int_{0}^{\frac{\pi}{2}} e^x \sin x\,dx$
1.3 $\displaystyle\int_{0}^{1} (3x+1)^{99}\,dx$	1.6 $\displaystyle\int_{0}^{1} \sqrt{\frac{4x^2-4x+1}{x^2-x+3}}\,dx$

2. Use First Mean Value Theorem for Integrals to prove the followingversion of the Mean Value Theorem for Derivatives. If $f \in C^1[a,b]$, then there is an $x_0 \in [a,b]$ such that
$$f(b)-f(a)=(b-a)f'(x_0).$$

3. If $f:[0,\infty)\to\mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dx}\int_0^{x^2} f(t)\,dt$.

4. If $g:\mathbb{R}\to\mathbb{R}$ is continuous, find $\displaystyle\frac{d}{dt}\int_{\cos t}^{t} g(x)\,dx$.

5. Let g be differentiable and integrable on $\mathbb{R}$. Define $f(x)=\displaystyle\int_1^{x^2} g(t)\cdot\sqrt{t}\,dt$.

   Show that $\displaystyle\int_0^1 xg(x)+f(x)\,dx=0$.

6. If $f(x)=\displaystyle\int_0^{x^2}\sec^2(t^2)dt$. show that $\displaystyle 2\int_0^1 \sec^2(x^2)\,dx-4\int_0^1 xf(x)\,dx=\tan 1$.

7. Suppose that g is integrable and nonnegative on $[1,3]$ with $\displaystyle\int_1^3 g(x)\,dt=1$. Prove that
$$\frac{1}{\pi}\int_1^9 g(\sqrt{x})\,dx<2.$$

8. Suppose that h is integrable and nonnegative on $[1,11]$ with $\displaystyle\int_1^{11} h(x)\,dt=3$. Prove that
$$\int_0^2 h(1+3x+3x^2-x^3)\,dx\le 1.$$

9. If f is continuous on $[a,b]$ and there exist numbers $\alpha\neq\beta$ such that
$$\alpha\int_a^c f(x)\,dx+\beta\int_c^b f(x)\,dx=0$$
holds for all $c\in(a,b)$, prove that $f(x)=0$ for all $x\in[a,b]$.

Chapter 8

Infinite Series of Real Numbers

8.1 Introduction

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of numbers. We shall call an expression of the form

$$\sum_{k=1}^{\infty} a_k$$

an **infinite series** with terms a_k.

Definition 8.1.1 *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series whose terms* a_k *belong to* $\mathbb{R}$*.*

1. *The* ***partial sums*** *of* S *of order* n *are the numbers defined, for each* $n \in \mathbb{N}$*, by*

$$s_n := \sum_{k=1}^{n} a_k.$$

2. S *is said to* ***converge*** *if and only if its sequence of partial sums* $\{s_n\}$ *to some* $s \in \mathbb{R}$ *as* $n \to \infty$*; i.e., for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$n \geq N \quad \textit{implies} \quad |s_n - s| < \varepsilon.$$

In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call s *the sum, or value, of the series* $\sum_{k=1}^{\infty} a_k$*.*

3. S *is said to* ***diverge*** *if and only if its sequence of partial sums* $\{s_n\}$ *does not converge.*

Example 8.1.2 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]=1.$

Example 8.1.3 *Prove that* $\displaystyle\sum_{k=1}^{\infty}(-1)^k$ *diverges.*

Theorem 8.1.4 (Harmonic Series) *Prove that the sequence* $\frac{1}{k}$ *converges but the series*

$$\sum_{k=1}^{\infty}\frac{1}{k} \quad \textit{diverges.}$$

Theorem 8.1.5 (**Divergence Test**) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *be a sequence of real numbers.*

$$\text{If } a_k \text{ does not converge to zero, then the series } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Example 8.1.6 *Show that the series* $\displaystyle\sum_{k=1}^{\infty} \frac{n}{n+1}$ *diverges.*

Theorem 8.1.7 (**Telescopic Seires**) *If* $\{a_k\}$ *is a convergent real sequence, then*

$$\sum_{k=m}^{\infty}(a_k - a_{k+1}) = a_m - \lim_{k\to\infty} a_k.$$

Example 8.1.8 *Evaluate the series* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}$.

Example 8.1.9 *Determine whether* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+1}+\sqrt{k}}$ *converges or not.*

Theorem 8.1.10 (**Geometric Seires**) *The series* $\sum_{k=1}^{\infty} x^k$ *converges if and only if* $|x| < 1$*, in which case*

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}.$$

Example 8.1.11 *Determine whether the following series converges or diverges.*

1. $\sum_{k=1}^{\infty} 2^{-k}$
2. $\sum_{k=1}^{\infty} (\sqrt{2}-1)^{-k}$

Theorem 8.1.12 *Let $\{a_k\}$ and $\{b_k\}$ be a real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then*

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \quad \text{and} \quad \sum_{k=1}^{\infty}(\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$

for any $\alpha \in \mathbb{R}$.

Theorem 8.1.13 *If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then*

$$\sum_{k=1}^{\infty}(a_k + b_k) \text{ diverges.}$$

Example 8.1.14 *Evaluate* $\displaystyle\sum_{k=1}^{\infty} \frac{1+2^{k+1}}{3^k}$.

Example 8.1.15 *Evaluate* $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.

Example 8.1.16 *Evaluate* $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{n(n+1)} + \frac{5^k}{2^k}\right)$.

Example 8.1.17 *Let* π *be a Pi constant. Show that*

$$\sum_{k=1}^{\infty} \frac{1}{\pi^{k^2}} \left[1 - \frac{\pi^{2k}}{\pi} + \left(\frac{\pi^k}{\pi} \right)^k \right]$$

converges and find its value.

Example 8.1.18 *Evaluate the series* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$.

Exercises 8.1

1. Show that
$$\sum_{k=n}^{\infty} x^k = \frac{x^n}{1-x}$$
for $|x| < 1$ and $n = 0, 1, 2, ...$

2. Prove that each of the following series converges and find its value.

2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}}$

2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\pi^k}$

2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}+4}{5^k}$

2.4 $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{7^{k-1}}$

2.5 $\displaystyle\sum_{k=0}^{\infty} 2^k e^{-k}$

2.6 $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{2^k}$

3. Represent each of the following series as a telescopic series and find its value.

3.1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$

3.2 $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k(k+2)}{(k+1)^2}\right)$

3.3 $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{\frac{\pi}{4}}\left(1-\left(\frac{\pi}{4}\right)^{j_k}\right)$, where $j_k = -\dfrac{1}{k(k+1)}$ for $k \in \mathbb{N}$

4. Find all $x \in \mathbb{R}$ for which
$$\sum_{k=1}^{\infty} 3(x^k - x^{k-1})(x^k + x^{k-1})$$
converges. For each such x, find the value of this series.

5. Prove that each of the following series diverges.

5.1 $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k^2}$

5.2 $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^k$

5.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

6. Prove that if $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

7. Let $\{b_k\}$ be a real sequence and $b \in \mathbb{R}$.

7.1 Suppose that there is an $N \in \mathbb{N}$ such that $|b - b_k| \le M$ for all $k \ge N$. Prove that

$$\left| nb - \sum_{k=1}^{n} b_k \right| \le \sum_{k=1}^{N} |b_k - b| + M(n - N)$$

for all $n > N$.

7.2 Prove that if $b_k \to b$ as $k \to \infty$, then

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \to b \quad \text{as} \quad n \to \infty.$$

7.3 Show that converse of 7.2 is false.

8. A series $\sum_{k=0}^{\infty} a_k$ is said to be **Cesàro summable** to $L \in \mathbb{R}$ if and only if

$$\sigma_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k$$

converges to L as $n \to \infty$.

8.1 Let $s_n = \sum_{k=0}^{\infty} a_k$. Prove that $\sigma_n = \dfrac{s_1 + s_2 + \cdots + s_n}{n}$ for each $n \in \mathbb{N}$.

8.2 Prove that if $a_k \in \mathbb{R}$ and $\sum_{k=0}^{\infty} a_k = L$ converges, then c is Cesàro summable to L.

8.3 Prove that $\sum_{k=0}^{\infty} (-1)^k$ is Cesàro summable to $\frac{1}{2}$; hence the converge of 8.2 is false.

8.4 **TAUBER.** Prove that if $a_k \ge 0$ for $k \in \mathbb{N}$ and $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L, then $\sum_{k=0}^{\infty} a_k = L$.

9. Suppose that $\{a_k\}$ is a decreasing sequence of real numbers. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $ka_k \to 0$ as $k \to \infty$.

10. Suppose that $a_k \ge 0$ for k large and $\sum_{k=0}^{\infty} \dfrac{a_k}{k}$ converges. Prove that $\lim_{j\to\infty} \sum_{k=1}^{\infty} \dfrac{a_k}{j+k} = 0$.

11. If and $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, prove that $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges.

8.2 Series with nonnegative terms

INTEGRAL TEST.

Theorem 8.2.1 (**Integral Test**) *Suppose that* $f : [1, \infty) \to \mathbb{R}$ *is positive and decreasing on* $[1, \infty)$. *Then* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if*

$$\lim_{n \to \infty} \int_1^n f(x)\, dx < \infty.$$

Example 8.2.2 *Use the Integral Test to prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

Example 8.2.3 *Show that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges.*

Example 8.2.4 *Show that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *converges.*

p-SERIES TEST.

Theorem 8.2.5 (p-Series Test) *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Example 8.2.6 *Find* $p \in \mathbb{R}$ *such that* $\displaystyle\sum_{k=1}^{\infty} k^{p^2-2}$ *converges.*

Example 8.2.7 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2^k}{k2^k}\right)$ *converges or not.*

COMPARISON TEST.

Theorem 8.2.8 *Suppose that* $a_k \geq 0$ *for* $k \geq N$. *Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if its sequence of partial sums* $\{s_n\}$ *is bounded, i.e., if and only if there exists a finite number* $M > 0$ *such that*

$$\left|\sum_{k=1}^{n} a_k\right| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Theorem 8.2.9 (**Comparison Test**) *Suppose that there is an* $M \in \mathbb{N}$ *such that*

$$0 \leq a_k \leq b_k \quad \text{for all } k \geq M.$$

1. *If* $\sum_{k=1}^{\infty} b_k < \infty$, *then* $\sum_{k=1}^{\infty} a_k < \infty$.
2. *If* $\sum_{k=1}^{\infty} a_k = \infty$, *then* $\sum_{k=1}^{\infty} b_k = \infty$.

Example 8.2.10 *Determine whether the following series converges or diverges.*

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+3^k}$

Example 8.2.11 *Determine whether* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln k}$ *converges or diverges.*

LIMIT COMPARISON TEST.

Theorem 8.2.12 (**Limit Comparison Test**) *Suppose that a_k and b_k are positive for lagre k and*

$$L := \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists as an extended real number.

1. *If $0 < L < \infty$, then $\sum_{k=1}^{\infty} b_k$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
3. *If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Example 8.2.13 *Use the Limit Comparison Test to prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ *converge.*

Example 8.2.14 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2k^4+k+3}$ *converges or diverges.*

Example 8.2.15 *Determine whether* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ *converges or diverges.*

Theorem 8.2.16 *Let* $a_k \to 0$ *as* $k \to \infty$*. Prove that*

$$\sum_{k=1}^{\infty} \sin|a_k| \text{ converges} \quad \text{if and only if} \quad \sum_{k=1}^{\infty} |a_k| \text{ converges.}$$

Exercises 8.2

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty} \frac{k-3}{k^3+k+1}$

1.2 $\displaystyle\sum_{k=1}^{\infty} \frac{k-1}{k2^k}$

1.3 $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^p}, \quad p>1$

1.4 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}3^{k-1}}$

1.5 $\displaystyle\sum_{k=1}^{\infty} \left(10+\frac{1}{k}\right) k^{-e}$

1.6 $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2-\sqrt{k}}{k^4-k^2+1}$

2. Prove that each of the following series diverges.

2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$

2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\ln^p(k+1)}, \quad p>0$

2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+2k+3}{k^3-2k^2+\sqrt{2}}$

2.4 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\ln^p k}, \quad p\le 1$

3. Use the Comparison Test to determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\ln k}{k}}$ converges or diverges.

4. Find all $p \ge 0$ such that the following series converges. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\ln^p(k+1)}$

5. If $a_k \ge 0$ is a bounded sequence, prove that $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$ converges for all $p>1$.

6. If $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges, prove that $\displaystyle\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$ converges for all $p \ge 0$. What happen if $p<0$?

7. Prove that if $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ coverge, then $\displaystyle\sum_{k=1}^{\infty} a_kb_k$ also converges.

8. Suppose tha $a, b \in \mathbb{R}$ satisfy $\frac{b}{a} \in \mathbb{R}\backslash\mathbb{Z}$. Find all $q>0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)q^k} \qquad \text{converges.}$$

9. Suppose that $a_k \to 0$. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\displaystyle\sum_{k=1}^{\infty}(a_{2k}+a_{2k+1})$ converges.

8.3 Absolute convergence

Theorem 8.3.1 (**Cauchy Criterion**) *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$, *there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \textit{imply} \quad \left|\sum_{k=n}^{m} a_k\right| < \varepsilon.$$

Corollary 8.3.2 *Let* $\{a_k\}$ *be a real sequence. Then the infinite series* $\sum_{k=1}^{\infty} a_k$ *converges if and only if for every* $\varepsilon > 0$, *there is an* $N \in \mathbb{N}$ *such that*

$$n \geq N \quad \textit{implies} \quad \left|\sum_{k=n}^{\infty} a_k\right| < \varepsilon.$$

ABSOLUTE CONVERGENCE.

Definition 8.3.3 *Let* $S = \sum_{k=1}^{\infty} a_k$ *be an infinite series.*

1. *S is said to **converge absolutely** if and only if* $\sum_{k=1}^{\infty} |a_k| < \infty$.
2. *S is said to **converge conditionally** if and only if S converges but not absolutely.*

Theorem 8.3.4 *A series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely if and only if for every* $\varepsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon.$$

Theorem 8.3.5 *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Example 8.3.6 *Prove that* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ *converges absolutely but* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *is not.*

LIMIT SUPREMUM.

Definition 8.3.7 *The supremum s of the set of adherent points of a sequence $\{x_k\}$ is called the* ***limit supremum*** *of $\{x_k\}$, denoted by $s := \limsup_{k\to\infty} x_k$, i.e.,*

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \sup\{x_k : k \geq n\}.$$

Example 8.3.8 *Evaluate limit supremum of the following sequences.*

1. $x_k = \frac{1}{k}$
2. $y_k = \frac{(-1)^k}{k}$
3. $z_k = 1 + (-1)^k$

Theorem 8.3.9 *Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence.*

1. *If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.*
2. *If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many k.*

Theorem 8.3.10 *Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence. If $x_k \to x$ as $k \to \infty$, then*

$$\limsup_{k\to\infty} x_k = x.$$

Example 8.3.11 *Evaluate limit supremum of $\left\{\dfrac{k}{k+1}\right\}$.*

ROOT TEST.

Theorem 8.3.12 (**Root Test**) *Let* $a_k \in \mathbb{R}$ *and* $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. *If* $r < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely.*
2. *If* $r > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Example 8.3.13 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{1+2k}\right)^k$ *converges absolutely.*

Example 8.3.14 *Prove that* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3+(-1)^k}{2}\right)^k$ *diverges.*

RATIO TEST.

Theorem 8.3.15 (**Ratio Test**) *Let $a_k \in \mathbb{R}$ with $a_k \neq 0$ for large k and suppose that*

$$r := \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists as an extended real number.

1. *If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.*
2. *If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Example 8.3.16 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$ *converges absolutely.*

Example 8.3.17 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ *diverges.*

Exercises 8.3

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ 1.2 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$ 1.3 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$ 1.4 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

2. Decide, using results convered so far in this chapter, which of the following series converge and which diverge.

2.1 $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{\pi^k}$ 2.4 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+3}\right)^{k}$ 2.7 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k!}{(k+2)!}\right)^{k^2}$

2.2 $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k}$ 2.5 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$ 2.8 $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3+(-1)^k}{3}\right)^{k}$

2.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k+3^k}$ 2.6 $\displaystyle\sum_{k=1}^{\infty} \left(\pi - \frac{1}{k}\right) k^{-1}$ 2.9 $\displaystyle\sum_{k=1}^{\infty} \frac{(1+(-1)^k)^k}{e^k}$

3. Define a_k recursively by $a_1 = 1$ and

$$a_k = (-1)^k \left(1 + k\sin\left(\frac{1}{k}\right)\right)^{-1} a_{k-1}, \quad k > 1.$$

Prove that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges absolutely.

4. Suppose that $a_k \geq 0$ and $\sqrt[k]{a_k} \to a$ as $k \to \infty$. Prove that $\displaystyle\sum_{k=1}^{\infty} a_k x^k$ converges absolutely for all $|x| < \frac{1}{a}$ if $a \neq 0$ and for all $x \in \mathbb{R}$ if $a = 0$.

5. For eachof the following, find all values of $p \in \mathbb{R}$ for which the given series converges absolutely.

5.1 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln^p k}$ 5.3 $\displaystyle\sum_{k=1}^{\infty} \frac{k^p}{p^k}$ 5.5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^{kp}k!}{k^k}$

5.2 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\ln^p k}$ 5.4 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}(k^p-1)}$ 5.6 $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k^{2p}+1} - k^p)$

6. Suppose that $a_{kj} \geq 0$ for $k, j \in \mathbb{N}$. Set

$$A_k = \sum_{j=1}^{\infty} a_{kj}$$

for each $k \in \mathbb{N}$, and suppose that $\sum_{k=1}^{\infty} A_k$ converges.

6.1 Prove that $\displaystyle\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kj} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right)$

6.2 Show that $\displaystyle\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kj} \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right)$

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Prove that $\sum_{k=1}^{\infty} |a_k|^p$ converges for all $p \geq 1$.

8. Suppose that $\sum_{k=1}^{\infty} a_k$ converges conditionally. Prove that $\sum_{k=1}^{\infty} k^p a_k$ diverges for all $p \geq 1$.

9. Let $a_n > 0$ for $n \in \mathbb{N}$. Set $b_1 = 0$, $b_2 = \ln\left(\frac{a_2}{a_1}\right)$, and

$$b_k = \ln\left(\frac{a_k}{a_{k-1}}\right) - \ln\left(\frac{a_{k-1}}{a_{k-2}}\right), \quad k = 3, 4, ...$$

9.1 Prove that $r = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ if exists and is positive, then

$$\lim_{n\to\infty} \ln(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left(1 - \frac{k-1}{n}\right) b_k = \sum_{k=1}^{\infty} b_k = \ln r.$$

9.2 Prove that if $a_n \in \mathbb{R}\backslash\{0\}$ and $\left|\frac{a_{n+1}}{a_n}\right| \to r$ as $n \to \infty$, for some $r > 0$, then $|a_n|^{\frac{1}{n}} \to r$ as $n \to \infty$.

8.4 Alternating series

Theorem 8.4.1 (**Abel's Formula**) *Let* $\{a_k\}_{k\in\mathbb{N}}$ *and* $\{b_k\}_{k\in\mathbb{N}}$ *be real sequences, and for each pair of integers* $n \geq m \geq 1$ *set*

$$A_{n,m} := \sum_{k=m}^{n} a_k.$$

Then

$$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$$

for all integers $n > m \geq 1$.

Theorem 8.4.2 (**Dirichilet's Test**) *Let* $\{a_k\}$ *and* $\{b_k\}$ *be sequences in* $\mathbb{R}$*. If the sequence of partial sums* $s_n = \sum_{k=1}^{n} a_k$ *is bounded and* $b_k \downarrow 0$ *as* $k \to \infty$*, then*

$$\sum_{k=1}^{n} a_k b_k \quad \textit{converges.}$$

Corollary 8.4.3 (Alternating Series Test (AST)) *If $a_k \downarrow 0$ as $k \to \infty$, then*

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \textit{converges.}$$

Moreover, if $\sum_{k=1}^{\infty} a_k$ *converges, then*

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \textit{converges conditionally.}$$

Example 8.4.4 *Prove that* $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally.*

Example 8.4.5 *Prove that* $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ *converges conditionally.*

Example 8.4.6 *Prove that* $S(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ *converges for each* $x \in \mathbb{R}$.

Exercises 8.4

1. Prove that each of the following series converges.

1.1 $\displaystyle\sum_{k=1}^{\infty}(-1)^k\left(\frac{\pi}{2}-\arctan k\right)$

1.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^2}{2^k}$

1.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}$

1.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^p}, \quad p>0$

1.5 $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^p}, \quad x\in\mathbb{R},\ p>0$

1.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\frac{2\cdot 4\cdots(2k)}{1\cdot 3\cdots(2k-1)}$

1.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\ln(e^k+1)}$

1.8 $\displaystyle\sum_{k=1}^{\infty}\frac{\arctan k}{4k^3-1}$

2. For each of the following, find all values $x\in\mathbb{R}$ for which the given series converges.

2.1 $\displaystyle\sum_{k=1}^{\infty}\frac{x^k}{k}$

2.2 $\displaystyle\sum_{k=1}^{\infty}\frac{x^{3k}}{2^k}$

2.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{\sqrt{k^2+1}}$

2.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(x+2)^k}{k\sqrt{k+1}}$

2.5 $\displaystyle\sum_{k=1}^{\infty}\frac{2^k(x+1)^k}{k!}$

2.6 $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k(x+3)}{\cos k}\right)^k$

3. Using any test coveredin this chapter, find out which of the following series converge absolutely, which converge conditionally, and which diverge.

3.1 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^3}{(k+1)!}$

3.2 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)(-3)\cdots(1-2k)}{1\cdot 4\cdots(3k-2)}$

3.3 $\displaystyle\sum_{k=1}^{\infty}\frac{(k+1)^k}{p^k k!}, \quad p>e$

3.4 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k+1}$

3.5 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k+1}}{\sqrt{k}k^k}$

3.6 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\sin k}{k!}$

3.7 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k^2+1}}$

3.8 $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\ln(k+2)}{k}$

4. **ABEL'S TEST.** Suppose that $\sum_{k=1}^{\infty} a_k$ converges and $b_k \downarrow b$ as $k \to \infty$. Prove that

$$\sum_{k=1}^{\infty} a_k b_k \quad \text{converges.}$$

5. Use Dirichilet's Test to prove that

$$S(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges for all $x \in \mathbb{R}$.

6. Prove that $\sum_{k=1}^{\infty} a_k \cos(kx)$ converges for every $x \in (0, 2\pi)$ and every $a_k \downarrow 0$.
What happens when $x = 0$?

7. Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Prove that if $b_k \uparrow \infty$ and $\sum_{k=1}^{\infty} a_k b_k$ converges, then

$$b_m \sum_{k=m}^{\infty} a_k \to 0 \quad \text{as} \quad m \to \infty.$$

Reference

Gerald B. Folland. (1999). **Real Analysis Modern Technique and Their Applications**. John Wiley & Sons, Inc., New York.

Halsey L. Royden and Prtrick M. Fitzpatrick. (2010). **Real Analysis** (Fourth Edition). Pearson Education, Inc. New Jersey.

Pual Glendinning. (2012). **Maths in minutes**. Quercus Editions Ltd, London, England.

William R. Wade. (2004). **An Introduction Analysis** (Third Edition). Pearson Education. Inc., New Jersey.

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