# Chapter 1

# Arithmetic and Fractions

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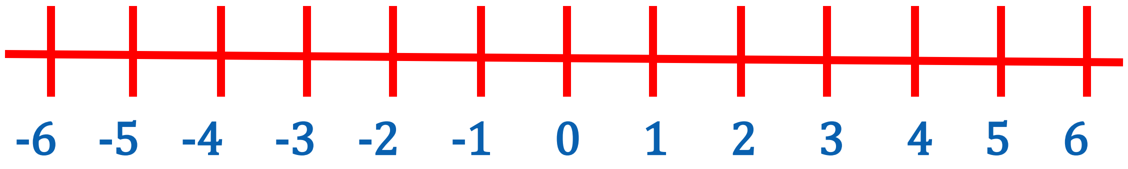
# 1.Properties of Real Numbers

When the test says "number," it always means a "REAL NUMBER"




A real number is any number on the number line. This includes round numbers, fractions, and decimal.

Zero is the only number that is neither positive nor negative

Integers include all positive and negative whole numbers, as well as zero:

{...-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...}

If the question asks about "integers," it could be any number in this set. If the question asks about "positive integers," then that's

{1, 2, 3, 4, 5, 6, 7, 8, 9, ...}

Which is the ordinary counting numbers.

## There are a few vocabulary words :

| | | |
|---|---|---|
| The result of addition | = | SUM |
| The result of subtraction | = | DIFFERENCE |
| The result of multiplication | = | PRODUCT |
| The result of division | = | QUOTIENT |

## Fundamental arithmetic properties common to all real numbers.

Let a, b, and c are real numbers.

1) Commutative Property : $a + b = b + a$ $a*b = b*a$

$2 + 3 = 3 + 2 = 5$ $3*5 = 5*3 = 15$

Addition and multiplication are commutative.

Division and subtraction, in general, are not commutative.

## 2) Associative Property

$$a + (b + c) = (a + b) + c$$

$$a * (b * c) = (a * b) * c$$

$$1 + (2 + 3) = (1 + 2) + 3 = 6$$

$$2 * (3 * 4) = (2 * 3) * 4 = 24$$

**Addition and multiplication are associative.**

**Division and subtraction, in general, are not associative.**

## 3) Distributive Property

$$a * (b + c) = a * b + a * c$$

or $$a * (b - c) = a * b - a * c$$

## 4) Multiplying and dividing by one

We don’t change a number when we multiply or divide by one:

$a * 1 = a$ $a/1 = a$

### Multiplying by zero

Anything times zero equals zero: $a * 0 = 0$ $0 * a = 0$

## 5) The Zero Product Property

If the product of two numbers is zero, one of the factors **must be** zero.

If a * b = 0 , then a = 0 **OR** b = 0

## 6) Dividing a number by itself

When any non-zero number is divided by itself, the quotient has to equal 1.

If $a \neq 0$ , then a/a = 1

# 2. Positive and Negative Numbers

***"Subtraction doesn't really exist" what does this mean?***

**Subtraction of any number can be re-written as addition of a number of the opposite sign.**

| | | |
|---|---|---|
| 30 - 18 | = | 30 + (-18) |
| (-24)-16 | = | (-24) + (-16) |
| 44 - (-14) | = | 44 + 14 |

## The Sign Rules for Multiplication :

(positive)*(positive) = positive

(negative)*(negative) = positive

(positive)*(negative) = negative

Same signs → positive product

Different signs → negative product

## The Sign Rules for Division :

(positive) / (positive) = positive

(negative) / (negative) = positive

(positive) / (negative) = negative

Same signs → positive quotient

Different signs → negative quotient

**Another topic related to positive and negative signs is absolute value. The definition of absolute value is "it makes everything positive"**

$$|5| = 5$$

$$|-9| = 9$$

$$|0| = 0$$

**A much more sophisticated definition is the absolute value of a number gives the distance of the number from the origin.**

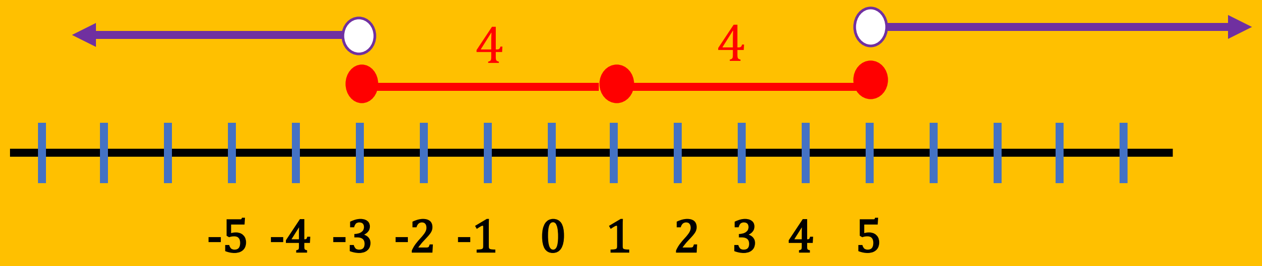
**For example**

$|a|$ = the distance of $a$ from the origin

$|a-9|$ = the distance of $a$ from 9

$|a+4|$ = the distance of $a$ from -4

Consider $|a - 1| > 4$




$\therefore a < -3 \; OR \; a > 5$

**Practice Problem: Consider the positive integers from 1-100. If $n$ is a number in that set, then for how many numbers $n$ is it true that $|n - 30| > 20$**

# 3. Mental Math, Addition, and Subtraction

1. **Practice mental math.**
2. **You can simplify mental math for addition of two digit numbers by treating the digit separately.**

**Ex**

42+17 = (40 + 2) + (10 + 7) = (40 + 10 ) + (2 +7) = 59

69+37 = (60+30) + (9 + 7) = 90 + 16 = 106

73+ 36 = ???

**3. You can also treat the digits separately in subtraction if the number subtracted has both digits smaller.**

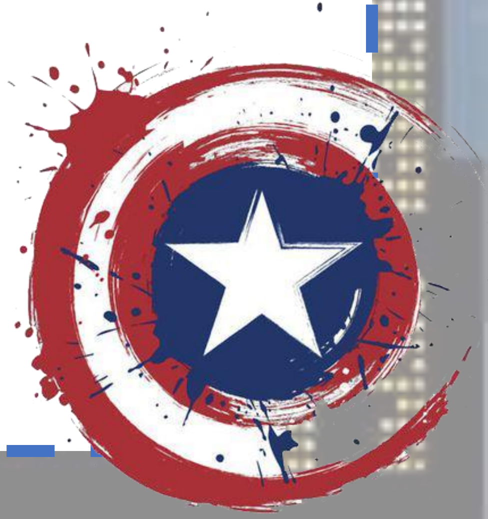
**Ex**

$$59 - 31 = (50 + 9) - (30 + 1)$$
$$= (50 - 30) + (9 - 1)$$
$$= 20 + 8 = 28$$
$$46 - 24 = ?$$
$$99 - 34 = ?$$

**4. You can also simplify subtraction by adding the same number to both terms.**

**Ex**

$$56 - 19 = (56+1) - (19+1) = 57 - 20 = 37$$
$$71 - 27 = ?$$
$$63 - 18 = ?$$



# 4. Order of Operations

**What is the PEMDAS?**

**The first ides we need to discuss is the idea of a mathematical grouping symbol. Any grouping symbol shows that certain operations are grouped together.**

1) Mathematical grouping symbols

**1.1 Parentheses ()**

$$(x+2)^2\ ,1/(x-y)$$

**1.2 Straight brackets [ ]**

$$[(x+3)^3-(x-1)^2]$$

***same meaning as parentheses***

**1.3 The square root sign**

$$\sqrt{15-x}$$

**1.4 The long fraction bar** → $\frac{x+3}{x-1}$

**1.5 The exponential "slot"** → $2^{x+4}$

2) Now we can talk about the Order of operations, many people call this PEMDAS but I will call it **GEMDAS.**

**2.1 Grouping symbols**

**2.2 Exponents**

**2.3 Multiplication & Division**

**2.4 Addition & Subtraction**

3) All M&D at the same level of priority

4) With nested grouping symbols, work from inside out

**For example,**

$6 + (8 - 3)^2 \times 4 \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$2 \times 4 + 7 \times \frac{12}{10-6} \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{60}{3\times(13-11)^2+8} + 2 \quad = \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

# 5. Decimals

## 1. Place value and decimals

First, the number **234** means.

| | |
|---|---|
| 2 | hundreds |
| 3 | tens |
| 4 | ones |

**Thus , 234 =** $2*(100) + 3*(10) + 4*(1)$

Notice that, as a digit is moved to the left, the place value is multiplied by 10

$$33 = 3.3 \times 10 \quad \text{or} \quad 33 = 0.33 \times 10^2$$

And as we move a digit to the right, the place value is divided by 10 such as

$$95 = 950 \times 10^{-1} \text{ or } \quad 123 = 12300 \times 10^{-2}$$

Consider the number 98.0675 has

9 in the **tens** place (10)
8 in the **ones** place (1)
0 in the **tenths** place (1/10)
6 in the **hundredths** place (1/100)
7 in the **thousandths** place (1/1,000)
5 in the **ten thousandths** place (1/10,000)

We could write this number as follows:

$$98.0657 = 90 \times 10 + 8 \times 1 + 0 \times \frac{1}{10} + 6 \times \frac{1}{100} + 7 \times \frac{1}{1000} + 5 \times \frac{1}{10000}$$

The decimal places to the right of the decimal point represent negative powers of ten:

0.1 = one tenths = $\frac{1}{10} = 10^{-1}$

0.01 = one hundredths = $\frac{1}{100} = 10^{-2}$

0.001 = one thousandths = $\frac{1}{1000} = 10^{-3}$

0.0001 = one ten thousandths = $\frac{1}{10000} = 10^{-4}$

Notice that additional zeroes to the right of any non-zero digits are meaningless: they make absolutely no difference in the value

3.78 = 3.7800 = 3.780000

Adding and subtracting decimals is quite straightforward. We simply line up the decimal points, and add or subtract vertically.

$2.81 + 3.647$ → 

$$\begin{array}{r} 2.18 \\ \underline{+\ 3.647} \\ 5.827 \end{array}$$

$1.111 – 0.05$ →

$$\begin{array}{r} 1.111 \\ \underline{-\ 0.05} \\ 1.061 \end{array}$$

For multiplication,

1. Counting the digit' numbers to the right of the decimal point. $6.25 \times 0.048$

2. Add those two: the product will have 2+3 = 5 decimal places

3) Ignore the decimal point entirely, and just find the product as if they were two positive integers. $625\times 48$= 30,000

4) We know we need five decimal places, so we put the five rightmost digits of this product to the right of the decimal. Thus

$6.25\times 0.048$ = 0.30000

## For example

**Step 1** $(0.02)^3 = 0.\underbrace{02}_{2} \times 0.\underbrace{02}_{2} \times 0.\underbrace{02}_{2}$

**Step 2** That is 2+2+2 $= 6$ places to the right of the decimal point.

**Step 3** $2^3 = 8$

**Step 4** So that rightmost 8 must land six places to the right of the decimal.

$$(0.02)^3 = 0.000008$$

## Division of decimals.

Just move the decimals to both numbers one place to the right, and continue unit the denominator, the divisor, is a whole number. we are multiplying the number by 10

$$\frac{1}{2.45} \times \frac{10}{10} = \frac{10}{24.5}$$

## For example

$$\frac{0.56}{0.0007} = \frac{5.6}{0.007} = \frac{56}{0.07} = 800$$

$$\frac{0.00013}{0.025} = \frac{0.0013}{0.25} = \frac{0.013 \times 4}{2.5 \times 4}$$

$$= \frac{0.0052}{1} = 0.0052$$

# 6. Rounding

Suppose we are rounding to the nearest integer, and suppose there is only one digit to the right of the decimal point.

The rule is very easy. If the number in the tenth's place is **0-4**, then we **round down**.
If the number in the tenth's is **5-9**, we **round up.**

| | | | |
|---|---|---|---|
| 74.9 | → 75 | 2.4 | → 2 |
| 37.5 | → 38 | 29.2 | → 29 |
| 394.0 | → 394 | 7.8 | → 8 |

Rounding to any decimal place: the nearest hundreds, the nearest thousandths, etc.

When you are asked to round to a place, look **only** at the digit in the next place to the right, i.e. the next smallest place. Same rule as above:

**0-4 --> round down**

**5-9 --> round up**

**For example,**

315.53**6** rounded to the nearest **hundredths** is 315.54

## For practice:

1493 rounded to the nearest tens place is …………………….

3.14158 rounded to the nearest ten thousandths place is ………..

3.14158 rounded to the nearest thousandths place is ……………….

3.14158 rounded to the nearest hundredths place is ………………..

3.14158 rounded to the nearest tenths place is …………….

69049 rounded to the nearest hundreds place is ………….

# 7. Fractions

## Fraction terms:

$$\frac{9}{15}$$

**The top part, the upstairs**, of a fraction is the **numerator.**

**This fraction has a numerator of 9.**

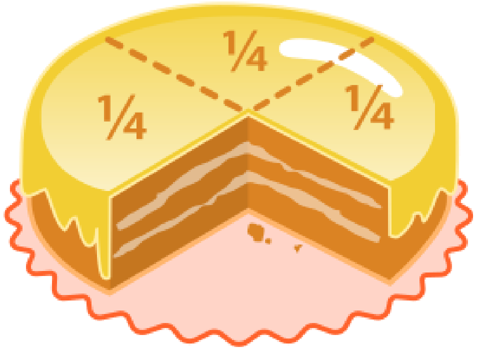
**The bottom part, the downstairs, of a fraction is the denominator.**

**This fraction has a denominator of 15.**

# Two ways of thinking about a fraction:

1) **Division –** 3/4 means 3 is divided by 4

2) **Pieces of cake –** if a "cake" is cut into four equal pieces, then 3/4 means three of those four pieces



Two fractions are equivalent if they have equal numerical value even though they have different numerators and different denominators.

$$\frac{2}{3}=\frac{4}{6}=\frac{2\times2}{3\times2} \quad , \quad \frac{3}{8}=\frac{3\times4}{8\times4}=\frac{12}{32}$$

As we can multiply both numerator and denominator by the same number, we can also **factor out** the same positive integer from both numerator and denominator.

$$\frac{6}{42}=\frac{6\times1}{6\times7}=\frac{1}{7}$$

The 6's **cancel.** Notice: canceling is a from of division

The equivalent fraction with the lowest possible integer values of numerator and denominator is the fraction written in **lowest terms.**

$$\frac{72}{96} = \frac{12}{16} = \frac{3}{4}$$

Lowest terms is also known as "simplest form"

If a fraction is greater than one, we can write the fraction as either:

1. an improper fraction (a fraction with numerator > denominator) such as $\frac{10}{3}$ , $\frac{7}{4}$.

Or

2. a mixed numeral (integer part + fraction part) such as $1\frac{2}{3}$

$$4\frac{3}{5} = 4 + \frac{3}{5} = 4 \times \frac{5}{5} + \frac{3}{5} = \frac{20}{5} + \frac{3}{5} = \frac{23}{5}$$

# 8. Conversions: Fractions and Decimals

Fraction from a group that mathematicians call the **Rational Numbers.** ("Rational" because they are "ratios.")

When we write a fraction, a rational number, as a decimal, one of two things happens: it either **terminates** or **repeats.**

**Terminating** decimals:

$\frac{1}{4} = 0.25$ $\frac{3}{8} = 0.375$

**Repeating** decimals:

$\frac{1}{3} = 0.333333\overline{3}$

$\frac{1}{7} = 0.142857\overline{142857}$

Another way to think about fractions with denominators that are multiples of 10

For example

$$\frac{1}{20} = \frac{1}{2} \times \frac{1}{10} = (0.5)(10^{-1}) = 0.05$$

$$\frac{1}{40} = \frac{1}{4} \times \frac{1}{10} = (0.25)(10^{-1}) = 0.025$$

$$\frac{1}{600} = \frac{1}{6} \times \frac{1}{100} = (0.1666\bar{6})(10^{-2}) = 0.001666\bar{6}$$

The decimals that neither terminate nor repeat are called the **irrational numbers.**

These decimals go on forever in a non-repeating pattern.

$$\pi = 3.14159265358 98 \ldots$$

$$\sqrt{2} = 3.4142135623731 \ldots$$

Golden Ratio

$$\frac{1+\sqrt{5}}{2} = 1.61803398874 99 \ldots$$

None of these can be written as a fraction of integer over integer.

# 9. Fraction Properties-I

1. Fractions with a denominator of 1 :

Let n is real number then n/1 = n

2. Fractions involving zero:

We cannot divide by zero. It is allowed to divide zero by a non-zero number.

0/5 = 0 0/(-3) = 0

Zero divided by any non-zero number equals zero.

3. As long as $n \neq 0$ , then $\frac{n}{n} = 1$

4. The reciprocal of a fraction, $a/b$, is the "flipped over" fraction, $b/a$. $(a \neq 0, b \neq 0)$

The reciprocal of $3/5$ is $5/3$

The reciprocal of $-7$ is $-1/7$

The reciprocal of $1/6$ is $6$

# 10. Comparing Fractions

1. With the <u>same denominator</u>, **bigger numerator** means a **bigger fraction**

$$\frac{1}{5} < \frac{2}{5} \qquad \frac{6}{10} < \frac{8}{10}$$

2. With the <u>same numerator</u>, **bigger denominator** means a **smaller fraction**

$$\frac{2}{10} > \frac{2}{11} \qquad \frac{1}{7} > \frac{1}{8}$$

What if we change both numerator and denominator?
If both the numerator gets bigger and the denominator gets smaller, then the fraction definitely gets bigger.

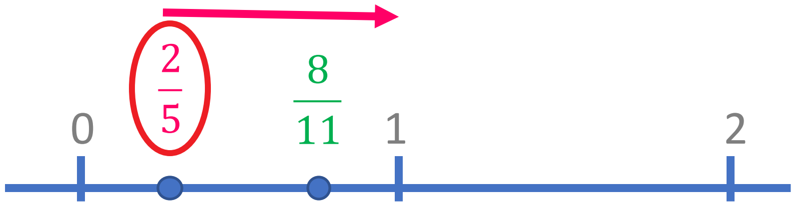
$$\mathbf{\frac{3}{8} < \frac{4}{7}}$$

3. Rules of adding numbers to the numerators and denominators.

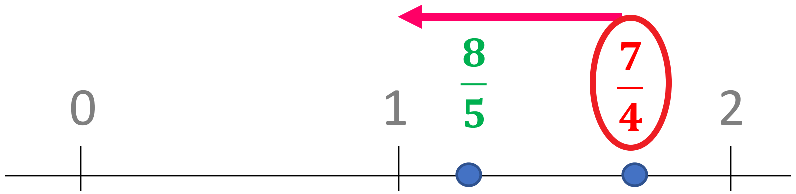
If we add the same number to both the numerator and the denominator, then the resultant fraction is closer to 1 than was the original fraction.

For example $\frac{2}{5}$, we add 6 to both the numerator and denominator.

$$\frac{2+6}{5+6} = \frac{8}{11}$$



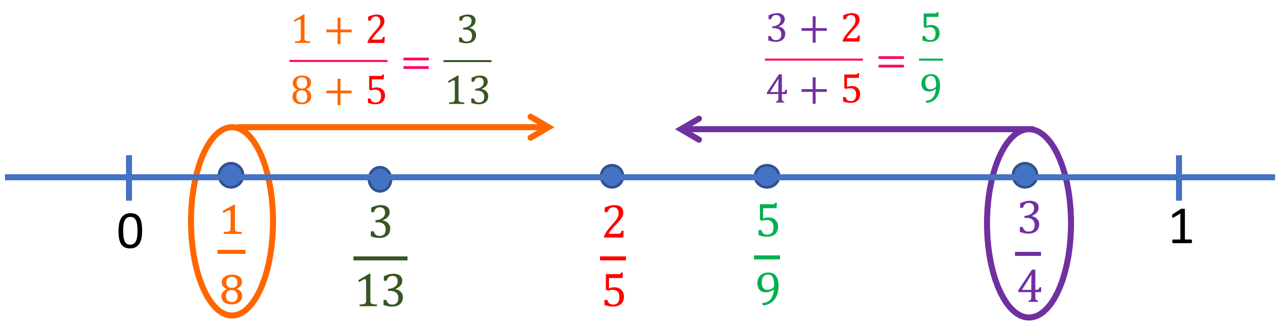

$$\frac{7}{4} \rightarrow \frac{7+1}{4+1} = \frac{8}{5}$$



If we add **something different** to the numerator and the denominator.

Suppose, we add 2 to the numerator and 5 to the denominator. The resultant fraction will be **closer to 2/5** on the number line than was the original fraction.

For example $\frac{3}{4}$ and $\frac{1}{8}$ ,we add 2 to the numerator and 5 to the denominator.



## Practice question

At certain office, in December 2019, there were 6 managers and 200 employees. On January 2, 2020, 1 manager and 30 employees joined the office. No one left the office.

| A | B |
| --- | --- |
| The manager-employee ratio in December, 2019. | The manager-employee ratio in late January, 2020. |

# 11. Operations with Fractions

1. With common denominators, we just add or subtract across the numerators.

Adding and subtracting fractions: finding a common denominator in case of the fraction do not have the same denominator.

$$\frac{1}{5} + \frac{3}{5} = \frac{4}{5} \qquad \frac{9}{13} - \frac{6}{13} = \frac{3}{13}$$

$$\frac{1}{5} + \frac{1}{10} = \frac{1}{5} \times \frac{2}{2} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$$

$$\frac{2}{3} - \frac{2}{4} = \frac{2}{3} \times \frac{4}{4} - \frac{2}{4} \times \frac{3}{3} = \frac{8}{12} - \frac{6}{12} = \frac{1}{6}$$

2. What's a little trickier about the multiplication of fraction is what you can cancel.

Cancel BEFORE you multiply

$$\frac{5}{14} \times \frac{7}{15} = \frac{1}{14} \times \frac{7}{3} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

3. Dividing with fractions, including a number divided by a fraction, a fraction divided by a number.

$$\frac{3/4}{5/6} = \frac{3}{4} \times \frac{6}{5} = \frac{9}{10} \qquad \frac{2}{5/2} = 2 \times \frac{2}{5} = \frac{4}{5} \qquad \frac{3/4}{2} = \frac{3}{4\times 2} = \frac{3}{8}$$

We can split fractions by add/subtract in the numerator, but **NOT** in the denominator.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{a+b}{c+d} = \frac{a}{c+d} + \frac{b}{c+d}$$

$$\frac{a}{c+d} \neq \frac{a}{c} + \frac{a}{d}$$

$$\frac{d}{e-f} \neq \frac{d}{e} - \frac{d}{f}$$

You may remember that an **improper fraction** is $\frac{17}{3}$, $\frac{40}{11}$

A mixed numeral expresses $5\frac{2}{3}$, $3\frac{7}{11}$

It's important to be comfortable changing back and forth between improper fractions and mixed numerals.

$$\frac{28}{5} = \frac{25}{5} + \frac{3}{5} = 5\frac{3}{5}, \qquad \frac{60}{7} = \frac{56}{7} + \frac{4}{7} = 8\frac{4}{7}$$

$$12\frac{1}{2} = \frac{24}{2} + \frac{1}{2} = \frac{25}{2}$$

# 12. Operations with Proportions

A **proportion** is an equation of the fraction = fraction

In this, we will simply discuss the rules: what we can do and CANNOT do with a proportion.

One big idea is **cross-multiplication.** When we have a proportion, we can immediately eliminate all fraction by cross multiplying.

$$\frac{a}{b} = \frac{c}{d}$$

$$ad = cb$$

For proportions with larger numbers, we should cancel before we cross-multiply.

BUT, be careful. What we **can** cancel in a proportion and what we **can't** cancel in a proportion is one of the most frequent problem areas of math. What exactly are we allowed to cancel in the proportion.

$$\frac{a}{b} = \frac{c}{d} \, ?$$

**We can cancel factors in the numerator and denominator of the same fraction (Vertical & Horizontal cancellation)**

$\frac{a}{b} = \frac{c}{d}$ Vertical $\frac{a}{b} = \frac{c}{d}$ Horizontal

**For example**

$\frac{25}{5} = \frac{1}{2x} \rightarrow \frac{5(5)}{5} = \frac{1}{2(x)}$ (Vertical cancellation)

$\frac{6}{x} = \frac{4}{3} \rightarrow \frac{2(3)}{x} = \frac{2(2)}{3}$ (Horizontal cancellation)

Don't $\frac{x}{2} = \frac{6}{5}$ -> $\frac{x}{\cancel{2}} = \frac{\cancel{2}(3)}{5}$ (we **cannot** do like this)

Diagonal cancellation

Practice solving these on your own:

$$\frac{12}{5x} = \frac{8}{15}$$

$$\frac{4x}{35} = \frac{10}{49}$$

$$\frac{96}{64} = \frac{54}{6x}$$

# 13. Word Problems with Fractions

As a general rule, when translating from words to math:

The word “ is ” means equals
The word “ of ” means multiply

For example, Bill’s monthly cable bill is 2/7 of his monthly rent. If he pays $300 on cable each month, what is his monthly rent?

Let C = cable , R = rent

$$C = \frac{2}{7}R$$

$$300 = \frac{2}{7}R$$

$$R = 300 \times \frac{7}{2} = \$\,1050$$

Chit's salary is 3/7 of June's salary and is 5/4 of Bank's salary. June's salary is what fraction of Bank's salary?

Let C= Chit’s salary , J=June’s salary and B = Bank’s salary
The question ask June’s salary is what fraction of Bank’s salary.

$$\therefore C = \frac{3}{7}J = \frac{5}{4}B$$

$$\frac{3}{7}J = \frac{5}{4}B \rightarrow J = \frac{7}{3} \times \frac{5}{4}B = \frac{35}{12}B$$

Thus, $\frac{35}{12}$ is the answer.