LINEAR ALGEBRA AND APPLICATIONS

Lecture Note by

Thanatyod Jampawai, Ph.D.

Suan Sunandha Rajabhat University, Version January, 2017

MAT2305

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Branch of Mathematics, Faculty of Education,

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Contents

Chapter 1

Systems of Linear Equations

Information in science and mathematics is often organized into rows and columns to form rectangular arrays, called **matrices** (plural of **matrix**). For example, we shall see this chapter that to solve a system of equations such as

$$\begin{cases} x + y &= 5 \\ x + 2y &= 7 \end{cases}$$

all of the information required for the solution is embodied in the matrix

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix}$$

and that the solution can be otained by performing approprate operations on matrix.

1.1 Linear Equations

Any straight line in the xy-plane can be represented algebraically by an equation of the form

$$ax + by = c$$

where a, b and c are real constants and a and b are not both zero. An equation of this form is called a linear equation in the variable x and y.

Definition 1.1.1: Linear Equation

We define a **linear equation** in the n variables $x_1, x_2, ., , , x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, ..., a_n$ and b are real constants. The variables in a linear equation are sometimes called **unknowns**.

Example 1.1.1 *Are equations linear ?*

1. $x - y = \sqrt{2}$
2. $x^2 + y = 6z$
3. $\sqrt{x} + 2y = 5$
4. $x_1 - 2x_3 - x_2x_3 = 1$
5. $\frac{1}{x} + y + z = 1$
6. $2x_1 + x_2 + x_3 = x_4$

Definition 1.1.2: Solution of Linear Equation

A **solution** of a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a sequence of n numbers $s_1, s_2, ..., s_n$ such that the equation is satisfied when we substitute

$$x_1 = s_1,\ x_2 = s_2, ..., x_n = s_n.$$

The set of all solutions of the equation is called its **solution set** or **general solution** of the equation.

Example 1.1.2 *Show that* $S = \{(x, y, z) : x = t - s, y = s, z = t\}$ *is a solution set of*

$$x + y - z = 1.$$

Example 1.1.3 *Find a linear equation in the variables* x *and* y *that has the general solution* $x = 1 - 3t$, $y = 2 + 5t$.

Example 1.1.4 *Finding solution sets.*

1. $x - y = 1$
2. $x + 2y = 4$
3. $x + 2y - z = 3$

Definition 1.1.3: Linear System

A finite set of linear equations of the variables $x_1, x_2, ..., x_n$ is called a **system of linear equation** or a **linear system**. A sequence of numbers $s_1, s_2, ..., s_n$ is called a **solution of the system** if the sequence is a solution of every equation in the system.

For example, the system

$$\begin{cases} x - y + z &= 3 \\ 2x + y - z &= 0 \end{cases}$$

has a solution $x = 1, y = 1, z = 3$ since these values satisfy both equation.

Example 1.1.5 *Show that* $S = \{(x, y, z) : x = 1, y = t, z = 2 + t\}$ *is a set of solution of linear system*

$$\begin{cases} x - y + z &= 3 \\ 2x + y - z &= 0 \end{cases}$$

Not all systems of linear equations have solution. For example,

$$\begin{cases} x + y & = 1 \\ 2x + 2y & = 6 \end{cases}$$

has no solution since the resulting equivalent system

$$\begin{cases} x + y & = 1 \\ x + y & = 3 \end{cases}$$

has contradictory equations.

Definition 1.1.4: Consistent

A system of equations that has no solution is said to be **inconsistent**; if there is at least one solution of the system, it is called **consistent**.

Example 1.1.6 *Dethermine whether the linear system is consistent or inconsistent.*

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 1 \\ x_1 - x_2 + x_3 - x_4 & = 3 \\ x_1 - x_2 - x_3 + x_4 & = -1 \end{cases}$$

Exercise 1.1

1. Finding solution sets.

   1.1 $x + y = 3$

   1.2 $x - 2y = 6$

   1.3 $2x + 3y = 12$

   1.4 $x + y - 3z = 2$

   1.5 $x_1 + x_2 = 3x_3 + 5$

   1.6 $2x_1 - 3x_3 = x_4 + x_5 - 7$

2. Find a linear equation in the variables x and y that has the general solution $x = 5 + 2t$, $y = t$.

3. Dethermine the linear systems are consistent or inconsistent and find their solution set.

   3.1 $\begin{cases} x - y &= 0 \\ x + y &= 2 \end{cases}$

   3.2 $\begin{cases} s - 2t &= 4 \\ 3s - 6t &= 12 \end{cases}$

   3.3 $\begin{cases} x - y + z &= 0 \\ x + y - z &= 3 \end{cases}$

   3.4 $\begin{cases} x - y &= 0 \\ x + y &= 2 \\ x + 2y &= 5 \end{cases}$

   3.5 $\begin{cases} x_1 - x_2 - x_3 + x_4 &= 7 \\ x_1 - 3x_2 &= -2 \end{cases}$

   3.6 $\begin{cases} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \end{cases}$

4. Prove that S is a set of solution of the following linear systems.

   4.1 $S = \{(x, y) : x = t, y = 4 - 2t\}$
   $$\begin{cases} 2x + y &= 4 \\ 4x + 2y &= 8 \end{cases}$$

   4.2 $S = \{(x, y, z) : x = 1 + t, y = t, z = 2t - 3\}$
   $$\begin{cases} x + y - z &= 4 \\ x - y &= 1 \end{cases}$$

   4.3 $S = \{(x_1, x_2, x_3) : x_1 = 1 + t, x_2 = 6 + t, x_3 = t\}$
   $$\begin{cases} x_1 + x_2 - 2x_3 &= 7 \\ x_1 - x_2 &= -5 \end{cases}$$

5. Find value(s) of a constant a such that $x = a + t, y = a + 2t, z = 1$ is a solution of linear system
$$\begin{cases} 2x - y &= 1 \\ 4x - 2y + z &= 3 \end{cases}$$

1.2 Gaussian Eliminations

In this section, we will study the systems of linear equation that can be written as matrix equations. We will look for a good algorithm for simplifying a system of linear equations or a matrix equation, called **Guassian elimination**. The linear system

$$\begin{cases} x+y &= 5 \\ x+2y &= 7 \end{cases}$$

is denoted by the **augmented matrix**,

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right].$$

Example 1.2.1 *Write the following linear equations by the augmented matrix.*

1. $\begin{cases} x-y &= 1 \\ 2x+y &= 2 \end{cases}$

2. $\begin{cases} 3x+y &= 0 \\ 2x+y &= -1 \\ x-y &= 2 \end{cases}$

3. $\begin{cases} 3x_1+x_2-x_3 &= 4 \\ x_1+2x_2+x_3 &= 5 \\ x_1-x_2 &= -3 \end{cases}$

In general case of r linear equations in k unknown, the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1k}x_k &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2k}x_k &= b_2 \\ &\vdots \\ a_{r1}x_1 + a_{r2}x_2 + ... + a_{rk}x_k &= b_r \end{aligned}$$

has an augmented matrix to be

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1k} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2k} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rk} & b_r \end{array}\right].$$

Consider the linear system

$$\begin{cases} x+y &= 5 \\ x+2y &= 7 \end{cases} \quad \leftrightarrow \quad \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right].$$

With the second equation replace by minus the first equation this becomes

$$\begin{cases} x+y &= 5 \\ 0+y &= 2 \end{cases} \quad \leftrightarrow \quad \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right].$$

Replace the new first equation by minus the second equation:

$$\begin{cases} x+0 &= 3 \\ 0+y &= 2 \end{cases} \quad \leftrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right].$$

To eliminate x from the second equation and then eliminate y from the first equation. The result was the solution of the linear system. Everywhere in the procedure above we can replace the word "equation" with the word "row" and interpret them as telling us what to do with the augmented matrix instead of the system of equations. Performed systemically, the result is the **Gaussian elimination** algorithm.

We introduce the symbol $\sim$ which read as "is equivalent to" because at each step the augmented matrix changes by an operation on its rows. For example, we found above that

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right].$$

This name **pivot** is used to indicate the matrix entry used to "zero out" the other entries in its column; the pivot is the number used to eliminate another number in its column.

Theorem 1.2.1

If two augmented matrices of both linear sysytems are equivalent then the thier solotions are same.

Example 1.2.2 *Find a system of linear equations corresponding to the augmented matrix.*

1. $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ -2 & 3 & 1 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 3 & 5 & 10 \\ 0 & -2 & 1 & -3 \end{array}\right]$

3. $\left[\begin{array}{cccc|c} -2 & 1 & -5 & 3 & 1 \\ 1 & 2 & 1 & -2 & -1 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$

Example 1.2.3 *Using Gussian elimination to solve the following systems of linear equations.*

1. $\begin{cases} x + y &= 3 \\ 2x - y &= 3 \end{cases}$

2. $\begin{cases} 3x + 2y &= -4 \\ 5x + y &= -9 \end{cases}$

3. $\begin{cases} 2x - 7y &= 1 \\ 7x - 2y &= -19 \end{cases}$

Example 1.2.4 *Find a solution of linear equations corresponding to the augmented matrix.*

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 2 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Exercise 1.2

1. Write the following linear equations by the augmented matrix and matrix equation.

1.1 $\begin{cases} x - y &= 3 \\ 2x + y &= 2 \end{cases}$

1.2 $\begin{cases} x + y - z &= 1 \\ x + 5y &= -4 \\ x - y + 2z &= 4 \end{cases}$

1.3 $\begin{cases} 5x + y - z + w &= -2 \\ 3x - y + z &= 1 \\ x - y - 5z - 3w &= 3 \end{cases}$

2. Find a system of linear equations corresponding to the argumented matrix.

2.1 $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 2 & 1 \end{array}\right]$

2.2 $\left[\begin{array}{ccc|c} 0 & 3 & -4 & 5 \\ 1 & 2 & 1 & -1 \end{array}\right]$

2.3 $\left[\begin{array}{cccc|c} 3 & 1 & 2 & 3 & 1 \\ 1 & 3 & 1 & 0 & -1 \end{array}\right]$

2.4 $\left[\begin{array}{cc|c} 5 & 9 & 6 \\ 13 & 2 & 1 \\ 2 & -3 & 0 \end{array}\right]$

2.5 $\left[\begin{array}{ccc|c} 4 & 3 & -1 & 7 \\ 8 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right]$

2.6 $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \end{array}\right]$

3. Using Gussian elimination to solve the following systems of linear equations.

3.1 $\begin{cases} x + y &= 7 \\ x - y &= 1 \end{cases}$

3.2 $\begin{cases} x + 2y &= 3 \\ x - 6y &= -5 \end{cases}$

3.3 $\begin{cases} 3x - y &= 5 \\ x + 2y &= -3 \end{cases}$

3.4 $\begin{cases} x + 6y &= 0 \\ 3x + 11y &= -7 \end{cases}$

3.5 $\begin{cases} x - 2y &= 5 \\ 5x - 10y &= 10 \end{cases}$

3.6 $\begin{cases} 3x - 2y &= 5 \\ 6x - 4y &= 10 \end{cases}$

4. Find a solution of linear equations corresponding to the argumented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

1.3 Elementary Row Operations (EROs)

The aim of Gaussian elimination for a system of three linear equations in the unknowns x, y and z is to convert the left part of the augmented matrix into the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right].$$

from which the solution $x = a, y = b$ and $z = c$ became evident. For many systems, it is not possible to reach this form. We need to perform operations that simplify our system without changing its solutions.

Definition 1.3.1: Elementary Row Operations (EROs)

We are lead to three operations:

1. **(Swap)** Exchange any two rows: R_{pq}.
2. **(Scaling)** Multiply any row by a non-zero constant: cR_q.
3. **(Replacement)** Add one scaling row to another row: $R_p + cR_q$.

These are called **Elementary Row Operations (EROs)**.

In any case, a certain version of the matrix that has the maximum number of components eliminated is said to be the **Row Reduced Echelon Form (RREF)**. To be of this form a matrix must have the following properties:

1. If a now does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **pivot** or **leading**.
2. If there are any rows that consist entirely of zeros, then they grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the pivot 1 in the lower row occurs farther to right than pivot 1 in the higher row.
4. Each column that contains a pivot 1 has zeros everwhere else.

A matrix that has first three properties (1-3) is siad to be in **row echelon form (REF)**.

Example 1.3.1 *The following augmented matrices in REF or RREF or NOT.*

1. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Example 1.3.2 *Suppose that augmented matrix for a linear system has been reduced by row operations to the give RREF. Solve the system.*

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

2. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Elimination Method. We have just seen how easy it is to solve a system of linear equations once its argumented matrix in RREF. Algorithm for obtaining RREF:

1. Make the leftmost nonzero entry in the top row 1 by multiplication.
2. Then use that 1 as a **pivot** (or **leading**) to eliminate everything below it.
3. Then go to the next row and make the leftmost nonzero entry 1.
4. Use that 1 as a pivot to eliminate everything below and above it.
5. Go to the next row and make the leftmost nonzero entry 1... etc

This procedure is called **Gauss-Jordan elimination**.

Example 1.3.3 *Solve by Gauss-Jordan elimination of a linear system for augmented matrix*

$$\left[\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 6 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & 5 & 1 \end{array}\right]$$

Example 1.3.4 *Compute solutions of the following linear systems by operating the augmented matrix in RREF.*

1. $$\begin{cases} x + y - z &= 1 \\ 2x + y - 3z &= 2 \\ y - 5z &= 6 \end{cases}$$

2. $$\begin{cases} x + y - z &= 2 \\ x - y + 2z &= 1 \\ 2x - z &= 1 \end{cases}$$

Example 1.3.5 *Solve by Gauss-Jordan elimination.*

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 & = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = -1 \\ 5x_3 + 10x_4 + 15x_6 & = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = 6 \end{cases}$$

Homogeneous Linear Equation. A system of linear equations is said to be homogeneous if the constant term all zero; that is, the system has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1k}x_k &= 0\\
a_{21}x_1 + a_{22}x_2 + ... + a_{2k}x_k &= 0\\
&\vdots\\
a_{r1}x_1 + a_{r2}x_2 + ... + a_{rk}x_k &= 0
\end{aligned}
$$

Every homogeneous system of linear equations is consistent, since all such system have

$$x_1 = 0, x_2 = 0, ..., x_n = 0$$

as a solution. This is called the **trivial solution**; if there are other solutions, they are called **nontrivial solution**.

Theorem 1.3.1

A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

Example 1.3.6 *Solve the following homogeneous system of linear equations by Solve by Guss-Jordan elimination.*

1. $\begin{cases} x + 3y - z &= 0 \\ 2x + y + z &= 0 \\ x - y + z &= 0 \end{cases}$

2. $$\begin{cases} x + 2y - z + w &= 0 \\ x + y + 2z + w &= 0 \\ 2x - z &= 0 \end{cases}$$

Example 1.3.7 *Solve the following nonlinear system.*

$$\begin{cases} 2 \cdot e^x - 1 \cdot e^y + 2 \cdot e^z & = 6 \\ 2 \cdot e^x + 2 \cdot e^y - 3 \cdot e^z & = -3 \\ 3 \cdot e^x - 2 \cdot e^y + 5 \cdot e^z & = 14 \end{cases}$$

Exercise 1.3

1. Which of the following matrices are in RREF ?

1.1 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ 1.3 $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ 1.5 $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ 1.7 $\left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

1.2 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ 1.4 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$ 1.6 $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ 1.8 $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

2. Which of the following matrices are in REF ?

2.1 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ 2.2 $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ 2.3 $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ 2.4 $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$

3. Suppose that argumented matrix for a linear system has been reduced by row operations to the give RREF. Solve the system.

3.1 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$ 3.2 $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ 3.3 $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

4. Solve each of the following system by Gauss-Jordan elimination.

4.1 $\begin{cases} x + y + 2z & = 8 \\ -x - 2y + 3z & = 1 \\ 3x - 7y + 4z & = 10 \end{cases}$ 4.2 $\begin{cases} x_1 + x_2 + x_3 & = 0 \\ -2x_1 + 5x_2 + 2x_3 & = 1 \\ 8x_1 + x_2 + 4x_3 & = -1 \end{cases}$

4.3 $\begin{cases} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{cases}$

4.4 $\begin{cases} 2x - 3y &= -2 \\ 2x + y &= 1 \\ 3x + 2y &= 1 \end{cases}$

4.5 $\begin{cases} 3x_1 + 2x_2 - x_3 &= -15 \\ 5x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 11 \\ -6x_1 - 4x_2 + 2x_3 &= 30 \end{cases}$

4.6 $\begin{cases} 10y - 4z + w &= 1 \\ x + 4y - z + w &= 2 \\ 3x + 2y + z + 2w &= 5 \\ -2x - 8y + 2z - 2w &= -4 \\ x - 6y + 3z &= 1 \end{cases}$

4.7 $\begin{cases} 2s - 3t + 4u - w &= 0 \\ 7s + t - 8u + 9w &= 0 \\ 2s + 8t + u - w &= 0 \end{cases}$

4.8 $\begin{cases} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ 3x_1 - x_2 + x_3 - x_4 &= 0 \end{cases}$

5. Solve each of the following system by any method.

5.1 $\begin{cases} 4x - 8y &= 12 \\ 3x - 6y &= 9 \\ -2x + 4y &= -6 \end{cases}$

5.2 $\begin{cases} x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + 3x_2 + 2x_3 &= 0 \\ x_1 + x_2 - 5x_3 &= 4 \end{cases}$

5.3 $\begin{cases} a - 2b + c &= 10 \\ 3a + 5b - 2c &= -5 \end{cases}$

5.4 $\begin{cases} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{cases}$

6. Solve the linear system $\begin{cases} 3x + 2y &= a \\ 2x - 3y &= b \end{cases}$, where a and b are constants.

7. For which value(s) of a will following system have no solutions? Exactly one solution? Infinite many solution?

$$\begin{cases} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{cases}$$

8. For which value(s) of λ does the system equations

$$\begin{cases} (\lambda - 3)x + y &= 0 \\ x + (\lambda - 3)y &= 0 \end{cases}$$

have nontrivial solution ?

9. Solve the following system of nonlinear equations for the unknown α, β and γ.

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{cases}$$

Chapter 2

Matrices

2.1 Matrices and Matrix Operations

A matrix is a **rectangular array** which occur in other context a well of numbers. For example, the following rectangular array with two rows and seven columns might describe the number of hours that Aj.Thanatyod Jampwai, faculty of education, Suan Sunandha Rajabhat University, spent lecture two sujects during a certain week for semester 2 academic year 2016:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun
MAT1202: Set Theory	3	3	0	0	0	0	0
MAT2305: Linear Algebra	0	0	3	0	3	0	0

If we suppose the headings, then we are left with the following rectangular array of number with two rows and seven columns called a **matrix**:

$$\begin{bmatrix} 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 \end{bmatrix}.$$

Definition 2.1.1: Matrix

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.

The **size** of a matrix is described in term of the number of rows and columns it contains. For example,

$$\begin{bmatrix} -1 & 3 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

is a martix that its size is 2 by 4 (written 2×4).

A matrix with only one column is called a **column matrix** (or a **column vector**), and a matrix with only one row is called a **row matrix** (or a **row vector**). For example,

$$\text{row matrix:} \quad \begin{bmatrix} 1 & 0 & 2 & 3 & 5 \end{bmatrix} \quad \text{column matrix:} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Hence, a general 3×5 matrix might be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The preceding matrix A can be written as

$$[a_{ij}]_{m\times n} \quad \text{or} \quad [a_{ij}].$$

A matrix A with n rows and n columns is called a **square matrix of order *n***, and entries $a_{11}, a_{22}, ..., a_{nn}$ are said to be on the **main diagonal** of A.

Definition 2.1.2: Equality of Matrix

Two matrices are defined to be **equal** if they have the same size and thier corresponding entries are eqaul.

Example 2.1.1 *Compute x and y if $A = B$.*

$$A = \begin{bmatrix} 1 & x+1 \\ 3 & 2 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & y+2 \\ 1-y & 2 \end{bmatrix}.$$

Definition 2.1.3: Sum and Difference of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same size.

- The **sum** $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A. That is
$$A + B = [a_{ij} + b_{ij}]$$
- The **difference** $A - B$ is the matrix obtained by substracting the entries of B from the corresponding entries of A. That is
$$A - B = [a_{ij} - b_{ij}]$$

Matrices of different sizes cannot be added or substracted.

Example 2.1.2 *Find $A + B$, $A - B$, $A + C$ and $B - C$.*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & -2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad C = \begin{bmatrix} 1 & 6 \\ 7 & 2 \end{bmatrix}.$$

Definition 2.1.4: Scalar Multiplication

Let $A = [a_{ij}]$ be any matrix and c be any scalar. Then the **product** cA is the matrix obtained by multuplying each entry of the matrix A by c. The matrix cA is said to be a **scalar multiple of** A.

$$cA = [ca_{ij}]$$

Example 2.1.3 *For the matrices*

$$A = \begin{bmatrix} 1 & -4 & 4 & 1 \\ 0 & 3 & -2 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 3 & 5 \\ 3 & 2 & -1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 5 & 0 & 9 & 5 \\ 7 & 2 & 0 & -3 \end{bmatrix}.$$

Compute $2A$, $3B - A$, $C - (A + 2B)$ *and* $2(A + B) - 3C$

Definition 2.1.5: Matrix Multiplication

Let $A = [a_{ij}]$ be an $m \times r$ matrix and $B = [b_{ij}]$ be an $r \times n$ matrix. Then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots b_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots b_{rn} \end{bmatrix},$$

the entry c_{ij} of **product** AB which is an $m \times n$ matrix $[c_{ij}]$ is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}.$$

Example 2.1.4 *Find AB, BA, BC, CB, CA and AC.*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 7 & 2 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad C = \begin{bmatrix} 0 & 6 & -1 \\ 5 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Theorem 2.1.1

Assuming that the sizes of the matrices are such that indicated operations can be performed, the following rules of matrix arithemetic are valid.

1. $A + B = B + A$ (Commutative law for addition)
2. $A + (B + C) = (A + B) + C$ (Associative law for addition)
3. $A(BC) = (AB)C$ (Commutative law for multiplication)
4. $A(B + C) = AB + AC$ (Left distributive law)
5. $(B + C)A = BA + CA$ (Right distributive law)

Theorem 2.1.2

Assuming that the sizes of the matrices are such that indicated operations can be performed, the following rules of matrix arithemetic are valid. For any scalars a and b,

1. $A(B - C) = AB - AC$ and $(B - C)A = BA - CA$
2. $a(B + C) = aB + aC$ and $a(B - C) = aB - aC$
3. $(a + b)C = aC + bC$ and $(a - b)C = aC - bC$
4. $(ab)C = a(bC) = b(aC)$ and $a(BC) = B(aC) = (aB)C$

Definition 2.1.6: Transpose

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the **traspose of** A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A.

$$A^T = [a_{ji}]$$

Example 2.1.5 *Find transposes of the following matrices.*

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 2 & 7 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 9 \\ 5 & 8 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Theorem 2.1.3

Assuming that the sizes of the matrices are such that indicated operations can be performed, the following rules of transpose of matrices are valid.

1. $(A+B)^T = A^T + B^T$ and $(A-B)^T = A^T - B^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^T = A$
4. $(aA)^T = a(A^T)$ where a is a scalar.

Definition 2.1.7: Zero Matrix

A **zero matrix** is a matrix whose entries are zero, denoted by 0.

Theorem 2.1.4

Assuming that the sizes of the matrices are such that indicated operations can be performed, the following rules of matrix arithemetic are valid.

1. $A + 0 = A = 0 + A$ (0 is the identity under addition of matrix)
2. $A + (-A) = 0 = (-A) + A$ (additive inverse of matrix)
3. $0 - A = -A$
4. $0A = 0 = A0$

Exercise 2.1

1. Solve the following matrix equation for a, b, c and d.

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

2. Consider the following matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix} \text{ and } E = \begin{bmatrix} 6 & 1 & 0 \\ -1 & 1 & 2 \\ 4 & 1 & 4 \end{bmatrix}$$

Compute the following (where possible)

2.1 $D + E$

2.2 $D - E$

2.3 $3A$

2.4 $-6D$

2.5 $2B - C$

2.6 $4E - 2D$

2.7 $3(D + 2E)$

2.8 $-(A - 2A)$

2.9 $(D + E)^T$

2.10 $2A^T D$

2.11 $(2D^T - 5E)^T$

2.12 $(C^T)B$

2.13 AB

2.14 DE

2.15 EDE^T

2.16 $(AC)^T$

2.17 $A^T D$

2.18 $DD + EE$

2.19 $(EA + DA)B$

2.20 $(AB)^T B + A$

3. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfies the stead condition.

3.1 $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

3.2 $a_{ij} = \begin{cases} 1 & \text{if } |i-j| > 1 \\ -1 & \text{if } |i-j| \geq 1 \end{cases}$

3.3 $a_{ij} = \begin{cases} 2^{i-j} & \text{if } i \geq j \\ 1 & \text{if } i < j \end{cases}$

3.4 $a_{ij} = 0$ if $i \neq j$

3.5 $a_{ij} = 0$ if $i < j$

3.6 $a_{ij} = 0$ if $i > j$

3.7 $a_{ij} = i + j$

3.8 $a_{ij} = ij$

3.9 $a_{ij} = i^j$

2.2 Inverse Matrix

Definition 2.2.1: Identity Matrix

An **identity matrix**, denoted by I, is a square matrix with 1's on the main doagonal and 0's off the main diagonal.

If it is important to emphasize the size, we shall write I_n for $n \times n$ identity matrix, such as

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.}$$

Example 2.2.1 *Let* $A = [a_{ij}]$ *be* 2×3 *matrix. Find* I_2A *and* AI_3.

Definition 2.2.2: Invertible Matrix

Let A be a square matrix. If a matrix B of the same size to A can be found such that

$$AB = I = BA,$$

then A is said to be **invertible** and B is called an **inverse** of A. If no such matrix B can be found, then A is said to be **singular**.

Example 2.2.2 *Show that* B *is an inverse of* A.

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

Theorem 2.2.1

If a matrix is invertible, then it has a unique inverse.

As a consequence of this important result, we can say of the inverse of an invertible matrix A and it is denoted by A^{-1}.

Theorem 2.2.2: Inverse of 2×2 Matrix

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.2.3 *Find A^{-1}, B^{-1}, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$ and $(AB)^{-1}$.*

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} -3 & -5 \\ 1 & 2 \end{bmatrix}$$

Theorem 2.2.3

Let A and B be square matrices of the same size. If A and B are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Definition 2.2.3: Power of Matrix

Let A be a square matrix. Then we define the nonnegative integer n power of A to be

- the nonnegative integer power of A to be

$$A^0 = I \quad \text{and} \quad A^n = \underbrace{AA\cdots A}_{n-factors} \quad \text{for } n > 0$$

- if A is invertible

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n-factors}$$

Theorem 2.2.4

Let A be invertible matrix and n and m be integers. Then

1. $A^mA^n = A^{m+n}$
2. $(A^n)^m = (A^m)^n = A^{nm}$

Theorem 2.2.5

Let A be invertible matrix and n be an integer. Then

1. $(A^{-1})^{-1} = A$
2. $(A^n)^{-1} = (A^{-1})^n$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $(kA)^{-1} = \frac{1}{k}A^{-1}$ where $k \in \mathbb{R}$ and $k \neq 0$

Example 2.2.4 *Find* A^2, $(A^T)^{-1}$, A^{-3} *and* $(2A)^{-4}$ *if* $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Definition 2.2.4: Polynomial Matrix

Let A be a square matrix. If

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is any polynomial, the we define

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Example 2.2.5 *Let* $p(x) = 2x^2 - 3x + 4$ *and* $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$. *Find* $p(A)$.

Definition 2.2.5: Elementary Matrix

An $n \times n$ matrix is called an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation (ERO).

Example 2.2.6 *Listed below are elementary matrices. What is a single ERO produce them ?*

1. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Example 2.2.7 *For matrices*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & -1 & 3 & 5 \\ 1 & 3 & 7 & 0 \end{bmatrix}$$

Compute EA.

Theorem 2.2.6

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performing on A.

If an ERO is applied to an identity matrix to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again.

Row operation on I that **produce** E	Row operation on E that **reproduce** I
Multiply row i by $c \neq 0$	Multiply row i by $\frac{1}{c}$
Interchang row i and j	Interchang row i and j
Add c times row i to row j	Add $-c$ times row i to row j

For example,

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Multiply } 2^{nd} \text{ row by } 5} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \xrightarrow{\text{Multiply } 2^{nd} \text{ row by } \frac{1}{5}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{interchange } 1^{st} \text{ row and } 2^{nd}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{interchange } 1^{st} \text{ row and } 2^{nd}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Add 7 times of } 2^{nd} \text{ row to } 1^{st} \text{ row}} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{Add } -7 \text{ times of } 2^{nd} \text{ row to } 1^{st} \text{ row}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem 2.2.7

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Theorem 2.2.8: Equivalent Statements

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. A is invertible.
2. The RREF of A from A is I_n.
3. A is expressible as a product of elementary matrices.

We estabish a method for determining the inverse of an $n \times n$ invertible matrix A. We can find elementary matrices $E_1, E_2, .., E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

Multiplying this equation on the right by A^{-1} yields

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

Example 2.2.8 *Find inverse of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Example 2.2.9 *Find inverse of* $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -4 & 5 \\ 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$

Example 2.2.10 *Find inverse of* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \end{bmatrix}$

Example 2.2.11 *Find inverse of* $A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 5 & 0 \\ 6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example 2.2.12 *Show that matrix* A *is not invertible (singular)* $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Exercise 2.2

1. Use theorem 2.2.2 to compute the inverses of the following matrices.

   1.1 $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ 1.2 $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ 1.3 $C = \begin{bmatrix} -2 & 9 \\ -1 & 5 \end{bmatrix}$ 1.4 $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

2. Use the matrices in 1. to compute

   2.1 $(2A)^{-1}$ 2.3 $A^{-1} + B^{-1}$ 2.5 $A^{-1}B^{-1}C^{-1}$ 2.7 $D^{-1}D^T$

   2.2 $(A+B)^{-1}$ 2.4 $(ABC)^{-1}$ 2.6 $C^{-1}B^{-1}A^{-1}$ 2.8 $D^{-1}BD$

3. In each part use the information to find A

   3.1 $A^{-1} = \begin{bmatrix} 3 & -1 \\ 3 & 5 \end{bmatrix}$ 3.3 $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

   3.2 $(2A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & 2 \end{bmatrix}$ 3.4 $(I + 2A)^{-1} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

4. Let A be a matrix $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. Compute A^2, A^{-3} and $A^2 - 2A + I$.

5. Let A be a matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$. In each part find $p(A)$.

   5.1 $p(x) = x + 3$ 5.2 $p(x) = x^2 + 5x + 6$ 5.3 $p(x) = x^3 - x + 2$

6. Find the inverses of

   6.1 $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ 6.2 $\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$

7. Show that if a square matrix A satisfies $A^2 - 2A + I = 0$, then $A^{-1} = 2I - A$

8. Which of the following are elementary matrices ?

8.1 $\begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$ 8.3 $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ 8.5 $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 8.7 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8.2 $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ 8.4 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 8.6 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ 8.8 $\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

9. Find a row operation that will restore the given elementary matrix to an identity matrix.

9.1 $\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ 9.2 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 9.3 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 9.4 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

10. Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices E_1, E_2, E_3 and E_4 such that

10.1 $E_1A = B$ 10.2 $E_2B = A$ 10.3 $E_3A = C$ 10.4 $E_4C = A$

11. Find inverse (if they exist) of matrices

11.1 $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$ 11.3 $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$ 11.5 $\begin{bmatrix} 0 & 1 & 7 \\ 1 & 3 & 3 \\ -2 & -5 & 1 \end{bmatrix}$

11.2 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ 11.4 $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix}$ 11.6 $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{bmatrix}$

12. Let k be a nonzero real number. Find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

13. Show that

$$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

14. Show that A and B are row equivalent, and find a sequence of elementary row operations that produces B from A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

2.3 Linear Systems with Invertibility

Marix multiplication has an important application to system of linear equations. Consider any linear system of m linear equation in n unknowns.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{12}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

We can replace the m equations in this system by product matrices.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{12} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate matrices by A, $\boldsymbol{x}$ and $\boldsymbol{b}$, respectively, the original system becomes to

$$A\boldsymbol{x} = \boldsymbol{b}$$

The matrix A is called the cofficient matrix of the system. The argumented matrix for the system is obatined by

$$[\,A \mid \boldsymbol{b}\,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Theorem 2.3.1

Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

The following theorem provides a new method for solving certain linear system.

Theorem 2.3.2

Let A be an $n \times n$ matrix and let $\boldsymbol{b}$ be an $n \times 1$ matrix. If the linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution, then, namely,

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Example 2.3.1 *Use theorem 2.3.2 to find the system of linear equations*

$$(a)\begin{cases} 3x_1 + 2x_2 &= 5 \\ 7x_1 + 5x_2 &= 3 \end{cases} \qquad \textit{and} \qquad (b)\begin{cases} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 11 \end{cases}$$

Linear systems with a common coefficient matrix: One is concerned with solving a seguence of systems

$$A\boldsymbol{x} = \boldsymbol{b_1}, \quad A\boldsymbol{x} = \boldsymbol{b_2}, \quad A\boldsymbol{x} = \boldsymbol{b_3}, ..., \quad A\boldsymbol{x} = \boldsymbol{b_k}$$

If A is invertible, then the solutions

$$\boldsymbol{x_1} = A^{-1}\boldsymbol{b_1}, \quad \boldsymbol{x_2} = A^{-1}\boldsymbol{b_2}, \quad \boldsymbol{x_3} = A^{-1}\boldsymbol{b_3}, ..., \quad \boldsymbol{x_k} = A^{-1}\boldsymbol{b_k}$$

can be obtained with one matrix inversion and k matrix multiplications. A more effient method is to form the matrix (argumented matrix)

$$[\ A \mid \boldsymbol{b_1} \mid \boldsymbol{b_2} \mid \cdots \mid \boldsymbol{b_k}\ ]$$

Example 2.3.2 *Use Gauss-Jodan elimination to find two systems of linear equations*

$$(a)\begin{cases} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{cases} \qquad (b)\begin{cases} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{cases}$$

Theorem 2.3.3

Let A be a square matrix.

1. If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

2. If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Theorem 2.3.4: Equivalent Statements

Let A be an $n \times n$ matrix. Then TFAE.

1. A is invertible.
2. $A\boldsymbol{x} = \mathbf{0}$ has only trivial solution.
3. The RREF of A is I_n.
4. A is expressible as a product of elementary matrices.
5. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.
6. $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.

Theorem 2.3.5

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Example 2.3.3 *What conditions must* a, b *and* c *satisfy in order for the system of equations*

$$\begin{cases} x_1 + x_2 + 2x_3 & = a \\ x_1 + x_3 & = b \\ 2x_1 + x_2 + 3x_3 & = c \end{cases}$$

to be consistent ?

Example 2.3.4 *What conditions must* a, b *and* c *satisfy in order for the system of equations*

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = a \\ 2x_1 + 5x_2 + 3x_3 & = b \\ x_1 + 8x_3 & = c \end{cases}$$

to be consistent ?

Exercise 2.3

1. Use theorem 2.3.2 to find the system of linear equations

1.1 $\begin{cases} 4x_1 - 3x_2 &= 5 \\ 3x_1 - 2x_2 &= 7 \end{cases}$

1.2 $\begin{cases} x_1 - 2x_2 &= 1 \\ 3x_1 + 2x_2 &= -3 \end{cases}$

1.3 $\begin{cases} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{cases}$

1.4 $\begin{cases} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{cases}$

1.5 $\begin{cases} x + y + z &= 4 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{cases}$

1.6 $\begin{cases} x_1 + 2x_2 + 3x_3 &= a \\ 2x_1 + 5x_2 + 5x_3 &= b \\ 3x_1 + 5x_2 + 8x_3 &= c \end{cases}$

1.7 $\begin{cases} -x_2 - 2x_3 - 3x_4 &= 0 \\ x_1 + x_2 + 4x_3 + 4x_4 &= 7 \\ x_1 + 3x_2 + 7x_3 + 9x_4 &= 4 \\ -x_1 - 2x_2 - 4x_3 - 6x_4 &= 6 \end{cases}$

2. Solve the linear system to find general solution

$$\begin{cases} x_1 + 2x_2 + x_3 &= a \\ x_1 - x_2 + x_3 &= b \\ x_1 + x_2 &= c \end{cases}$$

Use the resulting formulas to find the solution if

2.1 $a = -1, b = 3, c = 4$ 2.2 $a = b = c = 2$ 2.3 $a = -1, b = c = 5$

3. Solve the systems in both part at the same time.

(a) $\begin{cases} x_1 - 2x_2 + x_3 &= -2 \\ 2x_1 - 5x_2 + x_3 &= 1 \\ 3x_1 - 7x_2 + 2x_3 &= -1 \end{cases}$ (b) $\begin{cases} x_1 - 2x_2 + x_3 &= 1 \\ 2x_1 - 5x_2 + x_3 &= -1 \\ 3x_1 - 7x_2 + 2x_3 &= 0 \end{cases}$

4. Find conditions that b's must satisfy for the system to be consistent.

4.1 $\begin{cases} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{cases}$

4.2 $\begin{cases} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{cases}$

4.3 $\begin{cases} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{cases}$

4.4 $\begin{cases} x_1 - 2x_2 + x_3 &= b_1 \\ 2x_1 - 5x_2 + x_3 &= b_2 \\ 3x_1 - 7x_2 + 2x_3 &= b_3 \end{cases}$

5. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5.1 Show that the equation $A\boldsymbol{x} = \boldsymbol{x}$ can be rewritten as $(A - I)\boldsymbol{x} = \mathbf{0}$ and use this result to solve $A\boldsymbol{x} = \boldsymbol{x}$ for $\boldsymbol{x}$.

5.2 Solve $A\boldsymbol{x} = 4\boldsymbol{x}$.

6. Let $A\boldsymbol{x} = \boldsymbol{b}$ be any consistent system of linear equations, and let $\boldsymbol{x_1}$ be a fixed solution. Show that every solution to the system can be written in the form

$$\boldsymbol{x} = \boldsymbol{x_1} + \boldsymbol{x_0},$$

where $\boldsymbol{x_0}$ is a solution to $A\boldsymbol{x} = \mathbf{0}$. Show that also every matrix of this form is a solution.

2.4 Diagonal, Symmetric and Triangular Matrices

In this section we shall consider classes of matrices that have special forms.

Definition 2.4.1: Diagonal Matrix

A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**.

Here are some examples.

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Theorem 2.4.1

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero.

In this case the inverse of D is

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Power of diagonal matrices are easy to compute

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Example 2.4.1 *Let* A *be a matrix.* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ *Find* A^{-1}, A^3 *and* A^{-4}.

Example 2.4.2 *Let* D *be a* 3×3 *diagonal matrix and* $A = [a_{ij}]$ *be a* 3×3 *matrix. Compute* AD *and* DA.

Definition 2.4.2: Symmetric Matrix

A square matrix A is called **sysmmetric** if $A = A^T$.

The following matrices are symmetric.

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 6 & 2 & 1 \\ 2 & a & 0 \\ 1 & 0 & -5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 5 & 6 & 0 \\ 3 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.4.3 *Let A and B be matrices.*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \textit{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

Find A^T, $(A + B)$, $A - B$, $3A$, AB and A^{-1}.

Theorem 2.4.2

Let A and B be symmetric matrices with the same size and k be any scalar. Then

1. A^T is symmetric
2. $A+B$ and $A-B$ are symmetric
3. kA is symmetric

Theorem 2.4.3

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Example 2.4.4 *Let A be the 2×3 matrix* $A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$. *Find AA^T and A^TA.*

Theorem 2.4.4

If A is an invertible matrix, then A^TA and AA^T are invertible.

Definition 2.4.3: Lower and Upper triangular Matrices

A square matrix in which all the entries above the main giagonal are zero is called **lower triangular**, and a square matrix in which all the entries below the main giagonal are zero is called **upper triangular**. A matrix that is either upper triangular or lower triangular is called **triangular**.

Here are some examples of lower triangular matrices.

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 3 & a & 0 \\ 0 & k & b \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 3 & -8 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Here are some examples of upper triangular matrices.

$$\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} d & 5 & p \\ 0 & 3 & k \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 8 & 8 & 8 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2.4.5 *For upper triangular matrices,*

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Compute A^{-1} *and* AB.

Theorem 2.4.5

1. The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

2. The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

3. A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

4. The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Next, we will see how to write any square matrix A as the product of two simpler matrices. We will write

$$A = LU$$

where L is lower triangular and U is upper triangular. $A = LU$ is called an $\boldsymbol{LU}$ **decomposition** of A. If we give $A\boldsymbol{x} = \boldsymbol{b}$ and $A = LU$, then

$$LU\boldsymbol{x} = \boldsymbol{b}$$

we define $\boldsymbol{y} = U\boldsymbol{x}$ to solve for $\boldsymbol{x}$ and solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$.

Example 2.4.6 *Let A be a matrix such that $A = LU$.*

$$A = \begin{bmatrix} 4 & -2 & 4 \\ 2 & 0 & 3 \\ 0 & 3 & 4 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \quad \textit{and} \quad U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\textit{Solve system of linear equations} \begin{cases} 4x_1 - 2x_2 + 4x_3 & = 3 \\ 2x_1 + 3x_3 & = 4 \\ 3x_2 + 4x_3 & = 5 \end{cases}$$

Guassian elimination process can be stopped halfway to obtain decompositions. The first half of the elimination process is to eliminate entries below the diagonal leaving a matrix which is **upper triangular**. The elementary matrices which perform this part of the elimination are **lower triangular**. But putting together the upper triangular and lower triangular parts one obtains the so-called ***LU* factorization**.

Row operation on A that **produce** U	Row operation on I that **reproduce** L
Multiply row i by $c \neq 0$	Multiply row i by $\frac{1}{c}$
Interchang row i and j	Interchang row i and j
Add c times row i to row j	Add $-c$ times row i to row j

For example, matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

Consider row operations from A obtaining U

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \quad \overrightarrow{(-1)R_1 + R_3} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \overrightarrow{(-2)R_2 + R_3} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = U$$

Make row operations from I by reproducing prevous method to give L

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \overrightarrow{(+1)R_1 + R_3} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \overrightarrow{(+2)R_2 + R_3} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = L$$

Thus,

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = A.$$

Example 2.4.7 *Find a LU decomposition of the following matrices.*

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 2 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 3 & 6 \\ 3 & 2 & 5 \end{bmatrix}$$

Example 2.4.8 *Solve the linear system by LU decomposition.*

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = 5 \\ 2x_1 + 5x_2 + 3x_3 & = 8 \\ x_1 + 8x_3 & = 2 \end{cases}$$

Exercise 2.4

1. Find A^2, A^{-3} and A^k.

1.1 $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 1.2 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 1.3 $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. Which of the following matrices are symmetric ?

2.1 $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ 2.2 $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 6 & -3 \\ 4 & -3 & 5 \end{bmatrix}$ 2.3 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

3. Find all values of a, b and c for which A is symmetric.

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

4. Find all a and b for which A and B are both not invertible.

$$A = \begin{bmatrix} a+b-1 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 2a-3b-5 \end{bmatrix}$$

5. Find a diagonal matrix A that satisfies

5.1 $A^7 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ 5.2 $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ 5.3 $A^{-3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & -64 \end{bmatrix}$

6. A square matrix A is called **skew-symmetric** if $A^T = -A$.

6.1 Give example(s) skew-symmetric matrices.

6.2 Prove that if A and B are skew-symmetric with the same size, then are so A^T, $A+B$, $A-B$ and kA for any scalar k.

6.3 Prove that if A is an invertible skew-symmetric, then A^{-1} is skew-symmetric.

7. Let A be a symmetric matrix.

   7.1 Show that A^2 is symmetric.

   7.2 Show that $A^2 - 3A + I$ is symmetric.

8. Prove that if $A^T A = A$, then A is symmetric and $A^2 = A$.

9. Solve system of linear equations

   9.1 $$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

   9.2 $$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$$

10. Find a LU decomposition of the following matrices.

   10.1 $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & -5 \end{bmatrix}$

   10.2 $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$

   10.3 $C = \begin{bmatrix} -1 & 0 & 6 \\ 1 & 7 & 0 \\ 3 & -2 & -4 \end{bmatrix}$

11. Solve the linear systems by LU decomposition.

   11.1 $$\begin{cases} x_1 + x_2 + x_3 & = 5 \\ x_1 - 2x_2 + 3x_3 & = 0 \\ 2x_1 - x_3 & = 1 \end{cases}$$

   11.2 $$\begin{cases} -x_1 + 2x_2 - 2x_3 & = 5 \\ x_1 - 3x_2 + 3x_3 & = 6 \\ x_1 - 2x_2 - x_3 & = 10 \end{cases}$$

Chapter 3

Determinants

3.1 The Determinant Function

Definition 3.1.1: Permutation

A **permutation** of integers $\{1, 2, 3, ..., n\}$ is an arrangement of these integers in some order without omissions or repetitions.

For example, permutations of $\{1, 2, 3\}$ are

$$\begin{array}{ccc} (1,2,3) & (2,1,3) & (3,1,2) \\ (1,3,2) & (2,3,1) & (3,2,1) \end{array}$$

Example 3.1.1 *List out permutations of the set of integers* $\{1, 2, 3, 4\}$

We will denote a general permutation of the set $\{1, 2, ..., n\}$ by

$$(j_1, j_2, ..., j_n).$$

An **inversion** is said to be occur in a permutation $(j_1, j_2, ..., j_n)$ whenever a larger integer integer precedes a smaller one. For example, determine the number of inversion in the following permutations:

$$(a)\ (6, 1, 4, 3, 5, 2) \quad (b)\ (2, 4, 1, 3) \quad (c)\ (1, 2, 3, 4)$$

(a) The number of inversion is $5 + 0 + 3 + 1 + 1 = 10$

(b) The number of inversion is $1 + 2 + 0 = 3$

(c) There are no inversions in this permutation.

Example 3.1.2 *Find the number of inversions*

1. $(2, 1, 3, 4)$ *2.* $(4, 1, 3, 2)$ *3.* $(5, 1, 3, 2, 4)$ *4.* $(7, 5, 1, 4, 3, 6, 1)$

Definition 3.1.2: Even and Odd

A permutation is called **even** if the total number of inversions is an even integer and is called **odd** if the total number of inversions is an odd integer

The following table classifies the various permutation of $\{1, 2, 3\}$ as even or odd.

Permutation	**Number of inversions**	**Classification**
$(1, 2, 3)$	0	even
$(1, 3, 2)$	1	odd
$(2, 1, 3)$	1	odd
$(2, 3, 1)$	2	even
$(3, 1, 2)$	2	even
$(3, 2, 1)$	3	odd

Definition 3.1.3: Elementary Product

An **elementary product** of an $n \times n$ matrix A is any product of n entries from A, no two of which come from the same row or same column.

List all elementary product from the matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For A, elementary product can be written in the form

$$a_{1\square}a_{2\square}$$

where blanks $(\square, \square)$ designate column number from $\{1, 2\}$. Thus only elementary product are

$$a_{11}a_{22} \quad \text{and} \quad a_{12}a_{21}$$

For B, elementary product can be written in the form

$$a_{1\square}a_{2\square}a_{3\square}$$

where blanks $(\square, \square, \square)$ designate column number from $\{1, 2, 3\}$. Thus only elementary product are

$$\begin{array}{ccc} a_{11}a_{22}a_{33} & a_{12}a_{21}a_{33} & a_{13}a_{21}a_{32} \\ a_{11}a_{23}a_{32} & a_{12}a_{23}a_{31} & a_{13}a_{22}a_{31} \end{array}$$

Hence, an $n \times n$ matrix has $n!$ elementary products. They are the products of the form

$$a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

where $(j_1, j_2, ..., j_n)$ is a permutation of the set $\{1, 2, 3, ..., n\}$

Definition 3.1.4: Signed Elementary Product

Let $A = [a_{ij}]$ be an $n \times n$ matrix. A **signed elementary product from $\boldsymbol{A}$** means an elementary product $a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ multiplied by $+$ or $-$. We use

- the $+$ if $(j_1, j_2, ..., j_n)$ is an even permutation
- the $-$ if $(j_1, j_2, ..., j_n)$ is an odd permutation

For example, since $(3, 2, 1)$ is odd and $(1, 2, 3)$ is even, $+a_{11}a_{22}a_{33}$ and $-a_{13}a_{22}a_{31}$.

List all signed elementary product from the matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Elementary Product	**Associated Permutation**	**Number of Inversions**	**Classification**	**Signed Elementary Product**
$a_{11}a_{22}$	$(1,2)$	0	even	$+a_{11}a_{22}$
$a_{12}a_{21}$	$(2,1)$	1	odd	$-a_{12}a_{21}$

Elementary Product	**Associated Permutation**	**Number of Inversions**	**Classification**	**Signed Elementary Product**
$a_{11}a_{22}a_{33}$	$(1,2,3)$	0	even	$+a_{11}a_{22}a_{33}$
$a_{11}a_{23}a_{32}$	$(1,3,2)$	1	odd	$-a_{11}a_{23}a_{32}$
$a_{12}a_{21}a_{33}$	$(2,1,3)$	1	odd	$-a_{12}a_{21}a_{33}$
$a_{12}a_{23}a_{31}$	$(2,3,1)$	2	even	$+a_{12}a_{23}a_{31}$
$a_{13}a_{21}a_{32}$	$(3,1,2)$	2	even	$+a_{13}a_{21}a_{32}$
$a_{13}a_{22}a_{31}$	$(3,2,1)$	3	odd	$-a_{13}a_{22}a_{31}$

Definition 3.1.5: Determinant

Let A be a square matrix. The **determinant function** is denoted by ***det***, and we define $\det(A)$ or $|A|$ to be the sum of all signed elementary products from A. The number $\det(A)$ is called the **determinant of *A***.

Example 3.1.3 *Compute determinant of the matrices.*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example 3.1.4 *Use example 3.1.3 to compute determinant of the matrices.*

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 2 \\ 6 & 8 & 5 \end{bmatrix}$

Example 3.1.5 *Compute determinant*

$$\begin{vmatrix} a & a+1 & a+2 \\ a+3 & a+4 & a+5 \\ a+6 & a+7 & a+8 \end{vmatrix}$$

where a is any scalar.

Example 3.1.6 *Use the determinant definition to evaluate.*

1. $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

2. $$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Exercise 3.1

1. Find the number of inversions in each the following permuattions of $\{1, 2, 3, 4, 5\}$ and classify each permuattions as even or odd.

   1.1 $(4, 1, 3, 5, 2)$ 1.2 $(5, 4, 3, 1, 2)$ 1.3 $(2, 4, 3, 1, 5)$ 1.4 $(1, 4, 2, 3, 5)$

2. Evaluate the determinant.

   2.1 $\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix}$

   2.2 $\begin{vmatrix} 10 & 8 \\ -1 & 5 \end{vmatrix}$

   2.3 $\begin{vmatrix} 1 & 4 & 1 \\ 2 & 5 & 5 \\ 3 & 0 & -3 \end{vmatrix}$

   2.4 $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

   2.5 $\begin{vmatrix} 1 & 7 & -1 \\ 0 & 8 & 5 \\ 0 & 0 & 2 \end{vmatrix}$

   2.6 $\begin{vmatrix} 1 & 0 & 7 \\ 0 & 5 & 0 \\ 3 & 0 & 9 \end{vmatrix}$

   2.7 $\begin{vmatrix} 3 & 4 & 7 \\ a & -2 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

   2.8 $\begin{vmatrix} 1 & 0 & 5 \\ 1 & a & 2 \\ 1 & 2a & 10 \end{vmatrix}$

   2.9 $\begin{vmatrix} 2550 & 2551 & 2552 \\ 2553 & 2554 & 2555 \\ 2556 & 2557 & 2558 \end{vmatrix}$

3. Find all values of λ for which $\det(A) = 0$.

   3.1 $\begin{vmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{vmatrix}$

   3.2 $\begin{vmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{vmatrix}$

4. Use the determinant definition to evaluate.

   4.1 $\begin{vmatrix} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{vmatrix}$

   4.2 $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{vmatrix}$

5. Solve for x.

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

6. Show that the value of the seterminant

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix}$$

does not depend on θ.

7. Prove that the matrices

$$A = \begin{vmatrix} a & b \\ 0 & c \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} d & e \\ 0 & f \end{vmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

8. Let

$$A = \begin{bmatrix} a & b & c \\ d & d & f \\ x_1 & y_1 & z_1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ d & d & f \\ x_2 & y_2 & z_z \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & b & c \\ d & d & f \\ x_1+x_2 & y_1+y_2 & z_1+z_2 \end{bmatrix}.$$

Show that $\det(C) = \det(A) + \det(B)$.

3.2 Elementary Matrices and Determinant

We begin with a fundamental theorem that will lead us to an effient procedure for evaluating the determinant of a matrix of any order n.

Theorem 3.2.1

Let A be a square matrix.

1. If A has a row of zeros or column of zeros, then $\det(A) = 0$.
2. $\det(A^T) = \det(A)$.

Example 3.2.1 *Compute the determinants.*

1. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & -1 \\ 0 & 0 & 0 \end{vmatrix}$
2. $\begin{vmatrix} 1 & 0 & 3 \\ -1 & 0 & 5 \\ 5 & 0 & 2 \end{vmatrix}$
3. $\begin{vmatrix} 1984 & 1987 & 1990 \\ 1985 & 1988 & 1991 \\ 1986 & 1989 & 1992 \end{vmatrix}$

Theorem 3.2.2

Let $A = [a_{ij}]$ be an $n \times n$ matrix. If A is triangular, then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is,

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

The next theorem shows how an ERO on a matrix affects the value of its determinant.

Theorem 3.2.3

Let A be an $n \times n$ matrix.

1. If B is the matrix that results when two rows (or two columns) of A are interchanged, then
$$\det(B) = -\det(A).$$
2. If B is the matrix that results when a single row (or single column) of A is multiplied by a scalar k, then
$$\det(B) = k\det(A).$$
3. If B is the matrix that results when a multiple of one row of A is added to another row (or when a multiple of one column of A is added to another column), then
$$\det(B) = \det(A).$$

Example 3.2.2 *Let* $\det(A) = 5$ *such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Compute determinant of the following matrices.

1. $\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
2. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix}$
3. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 3a_{11} & a_{22} - 3a_{12} & a_{23} - 3a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
4. $\begin{bmatrix} a_{12} & a_{11} & a_{23} \\ a_{22} & a_{21} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$
5. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 4a_{31} & 4a_{32} & 4a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$
6. $\begin{bmatrix} -2a_{21} & -2a_{22} & -2a_{23} \\ 3a_{11} + a_{31} & 3a_{12} + a_{32} & 3a_{13} + a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Theorem 3.2.4

Let E be an $n \times n$ elementary matrix.

1. If E results from multiplying a row of I_n, then $\det(E) = k$.
2. If E results from interchanging two rows of I_n, then $\det(E) = -1$.
3. If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

Example 3.2.3 *Compute determinant of the following elementary matrices.*

1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

If a square matrix A has two proportional rows, the row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. We must have $\det(A) = 0$

Theorem 3.2.5

If A is a matrix with two proportional rows or two proposional columns, then

$$\det(A) = 0.$$

Example 3.2.4 *Compute determinant of the following elementary matrices.*

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 2 & 4 & 6 \end{bmatrix}$
2. $\begin{bmatrix} 6 & x & -1 & 2 \\ 0 & 1 & -3 & 1 \\ 1 & 2 & 6 & -5 \\ 0 & -3 & 9 & -3 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 4 & 0 & -1 \\ 1 & -8 & 3 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -4 & 2 & 1 \end{bmatrix}$

Using Row Operation to Evaluate a Determinant

This table is relationship and operation for 3×3 matrix

Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	The first row is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	The first and second rows are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	A multiple of the second row is added to the first row.

Example 3.2.5 *Use row operation to evaluate* $\det(A)$ *and* $\det(B)$ *where*

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{vmatrix}$$

Exercise 3.2

1. Compute the following determinants

1.1 $\begin{bmatrix} 2 & -12 & 1 \\ 0 & 5 & -9 \\ 0 & 0 & 2 \end{bmatrix}$ 1.2 $\begin{bmatrix} 3 & 0 & 0 \\ 4 & -5 & 0 \\ 6 & 1 & 3 \end{bmatrix}$ 1.3 $\begin{bmatrix} 2 & -1 & 3 \\ 7 & 5 & -9 \\ 4 & -2 & 6 \end{bmatrix}$ 1.4 $\begin{bmatrix} 2 & -1 & -4 & 0 \\ -1 & 5 & 2 & 0 \\ -1 & -2 & 2 & 1 \\ 2 & -2 & -4 & 0 \end{bmatrix}$

2. Find the determinants of the following elementary matrices.

2.1 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ 2.2 $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2.3 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ 2.4 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. Evaluate the determinants of the matrix by reducing the matrix to row echelon form.

3.1 $\begin{bmatrix} 3 & 6 & -9 \\ 0 & 0 & 2 \\ -5 & 2 & 1 \end{bmatrix}$ 3.3 $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & -2 \end{bmatrix}$ 3.5 $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & 2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$ 3.7 $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

3.2 $\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$ 3.4 $\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}$ 3.6 $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ 3.8 $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$

4. Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -3$, find

4.1 $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ -g & -h & -i \end{vmatrix}$ 4.2 $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ 4.3 $\begin{vmatrix} a+g & b+h & c+i \\ g & h & i \\ 3d & 3e & 3f \end{vmatrix}$ 4.4 $\begin{vmatrix} 4a & 4b & 4c \\ d & e & f \\ 5g-d & 5h-e & 5i-f \end{vmatrix}$

3.3 Properties of the Determinant

In this section, we develope some basic properties of determinant function. Suppose that A and B are $n \times n$ matrices and k is any scalar. We begin by considering possible relationships between $\det(A)$ and $\det(B)$.

$$\det(A+B), \quad \det(kA) \quad \text{and} \quad \det(AB).$$

Theorem 3.3.1: Additional Determinant

Let A, B and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then

$$\det(C) = \det(A) + \det(B).$$

Example 3.3.1 *By evaluating the determinant, check*

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Theorem 3.3.2: Scalar Multiplication to Determinant

Let A be an $n \times n$ matrix and k be any scalar. Then

$$\det(kA) = k^n \det(A).$$

Example 3.3.2 *Evaluate the determinant of the matrix*

$$\det(2A),\ \det(-A),\ \det(3A^T) \quad \textit{and} \quad \det(-(-2A)^T)$$

if $\det(A) = -5$ *for* 3×3 *matrix* A.

Theorem 3.3.3

If B is an $n \times n$ matrix and E is $n \times n$ elementary matrix, then

$$\det(EB) = \det(B)\det(B).$$

Moreover, if $E_1, E_2, ..., E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B).$$

Theorem 3.3.4: Invertible Test

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Example 3.3.3 *Use theorem 3.3.4 to determine which of the following matrices are invertible.*

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & 7 \\ 2 & 4 & 6 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Theorem 3.3.5: Product Determinant

If A and B are square matrix of the same size, then

$$\det(AB) = \det(A)\det(B).$$

If m is a positive integer,

$$\det(A^m) = [\det(A)]^m.$$

Example 3.3.4 *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 2 \\ 6 & -1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 11 & 4 \\ 5 & 1 \end{bmatrix}$. *Find*

1. $\det(AB)$
2. $\det(BC)$
3. $\det(A^T C)$
4. $\det(2AB)$
5. $\det(-3AB^T C)$
6. $\det(2C(AB^2)^T)$
7. $\det(A^2 B^3 C^2)$
8. $\det(A + A^T)$

Example 3.3.5 *Find* $\det(A)$ *satisfying*

1. $5A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$
2. $A^2 = \begin{bmatrix} 2 & -2 \\ 3 & 5 \end{bmatrix}$
3. $-7A^T = \begin{bmatrix} 6 & 2 \\ 13 & 5 \end{bmatrix}$

The following theorem gives a useful relationship between the determinant of an invertible matrix and determinant of its inverse.

Theorem 3.3.6: Determinant of Inverse

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example 3.3.6 *Let* $A = \begin{bmatrix} 6 & -4 \\ -3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 8 & 4 \\ 5 & 1 \end{bmatrix}$. *Find*

1. $\det(A^{-1})$
2. $\det(AB^{-1})$
3. $\det(3(-2A^T)^{-1})$
4. $\det(2(-3A)^T A^{-1} B^T)$

Theorem 3.3.7: Equivalent Statements

Let A be an $n \times n$ matrix. Then TFAE.

1. A is invertible.
2. $A\boldsymbol{x} = \mathbf{0}$ has only trivial solution.
3. The RREF of A is I_n.
4. A is expressible as a product of elementary matrices.
5. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.
6. $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.
7. $\det(A) \neq 0$.

Exercise 3.3

1. Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$. Find

   1.1 $\det(AB)$

   1.2 $\det(2BC)$

   1.3 $\det(-B^2)$

   1.4 $\det(-A^2B^{-2})$

   1.5 $\det(C^TA^{-1}B^T)$

   1.6 $\det(A^{-1}B)$

   1.7 $\det((2AC)^{-2})$

   1.8 $\det(-C^2B^3A)$

   1.9 $\det(-8(3A^{-2})C^{-1})$

2. By inspection, explain why $\det(A) = 0$ where

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

3. Use theorem 3.3.4 to determine which of the following matrices are invertible.

   3.1 $\begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$

   3.2 $\begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$

   3.3 $\begin{bmatrix} -3 & 0 & 1 \\ 3 & 0 & 6 \\ 5 & 0 & 7 \end{bmatrix}$

4. Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a+g & b+h & c+i \end{bmatrix}$$

   Assume that $\det(B) = -4$. Find

   4.1 $\det(A)$

   4.2 $\det(3A^{-1})$

   4.3 $\det(-B^{-2})$

   4.4 $\det(2A^TB^{-1})$

5. Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

6. Prove that a square matrix A is invertible if and only if A^TA is invertible.

3.4 Adjoint of a Matrix

Definition 3.4.1: Minor and Cofactor

Let A be a square matrix.

- The **minor** of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A.
- The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor** of entry a_{ij},

$$C_{ij} = (-1)^{i+j}M_{ij}$$

Example 3.4.1 *Find all minors and all cofactors of matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Theorem 3.4.1: Expansions by Cofactors

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be computed by

- cofactor expansion along the jth column: $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$
- cofactor expansion along the ith row: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

for each $1 \leq i \leq n$ and $1 \leq j \leq n$.

Example 3.4.2 *Find* $\det(A)$ *in example 3.4.1 by cofactor expansions.*

Example 3.4.3 *Find determinant of the following matrices by row opreations or cofactor expansions or both.*

1. $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 2 \\ 0 & 3 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 5 & 1 & 5 \\ 0 & 0 & -1 & 1 \\ 2 & 4 & 0 & 5 \\ -1 & 0 & 9 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 0 & 5 \\ 3 & 4 & 5 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$

Definition 3.4.2: Adjoint Matrix

Let A be any $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. The the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactor** from A. The transpose of this matrix is called the **adjoint** of A and denoted by $\text{adj}(A)$.

Example 3.4.4 *Find the matrix of cofactor and adjoint of the matrix A in example 3.4.1.*

Theorem 3.4.2: Inverse of Matrix Using its Adjoint

If A is invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

Example 3.4.5 *Use adjoint to find the inverse of the matrix A in example 3.4.1.*

Example 3.4.6 *Use adjoint to find the inverse of the matrix A where*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ 6 & 3 & -1 \end{bmatrix}$$

Theorem 3.4.3: Entry of Inverse Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. If A is invertible and $A^{-1} = [a^*_{ij}]$, then

$$a^*_{ij} = \frac{1}{\det(A)} C_{ji}$$

Example 3.4.7 *Use theorem 3.4.3 to compute entry of the inverse of matrix A where*

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{bmatrix}$$

If $A^{-1} = [a^*_{ij}]$, find $\quad a^*_{12} \quad$ *and* $\quad a^*_{42}$

Exercise 3.4

1. Find all minors and all cofactors of the matrices.

1.1 $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & 1 \\ 5 & 3 & 4 \end{bmatrix}$

1.2 $B = \begin{bmatrix} 2 & 1 & 9 \\ 2 & 3 & 1 \\ -3 & 5 & 0 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 9 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ Find

2.1 M_{13} and C_{13}

2.2 M_{24} and C_{24}

2.3 M_{32} and C_{32}

2.4 M_{41} and C_{41}

3. Evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

3.1 $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

3.2 $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

3.3 $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

3.4 $A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

3.5 $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

3.6 $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

4. Find inverse of A by its adjoint.

4.1 $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

4.2 $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$

4.3 $A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{bmatrix}$

4.4 $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & -3 & 3 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ If $A^{-1} = [a^*_{ij}]$, find

5.1 a^*_{11}

5.2 a^*_{41}

5.3 a^*_{13}

5.4 a^*_{24}

6. Let A and B be $n \times n$ matrices. For any scalar k, prove that

6.1 $\det(\mathrm{adj}(A)) = [\det(A)]^{n-1}$, $\det(A) \neq 0$

6.2 $\mathrm{adj}(AB) = \mathrm{adj}(BA)$

6.3 $A\mathrm{adj}(A) = \mathrm{adj}(A)A$

6.4 $\mathrm{adj}(kA) = k^n\mathrm{adj}(A)$

3.5 Cramer's Rule

The next theorem provides a formula for the solution of certain linear systems of n equations in n unknowns.

Theorem 3.5.1: Cramer's Rule

Let a system of n equations in n unknowns be

$$A\boldsymbol{x} = \boldsymbol{b}$$

with $\det(A) \neq 0$. Then the system has a unique solution and this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, ..., \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entrices in the jth column of A by the entries in the matrix $\boldsymbol{b}$.

Example 3.5.1 *Use Cramer's rule to solve*

1. $$\begin{cases} x_1 + x_3 &= 4 \\ x_1 + x_2 - x_3 &= -2 \\ 2x_1 - x_2 + x_3 &= 5 \end{cases}$$

2. $$\begin{cases} x + y + 2z &= 1 \\ 2x - y - 3z &= -8 \\ x - y + z &= 2 \end{cases}$$

Exercise 3.5

1. Use Cramer's rule to solve

1.1 $$\begin{cases} x_1 + x_3 &= 1 \\ x_1 + 2x_2 &= 2 \end{cases}$$

1.2 $$\begin{cases} 4x_1 + 5x_2 &= 2 \\ 11x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 5x_2 + 2x_3 &= 1 \end{cases}$$

1.3 $$\begin{cases} x - 4y + z &= 6 \\ 4x - y + 2z &= -1 \\ 2x + 2y - 3z &= -20 \end{cases}$$

1.4 $$\begin{cases} x - 3y + z &= 4 \\ 2x - y &= -2 \\ 4x - 3z &= 0 \end{cases}$$

1.5 $$\begin{cases} 3x_1 + x_2 - x_3 &= 4 \\ -x_1 + 7x_2 - 2x_3 &= 1 \\ 2x_1 + 6x_2 - x_3 &= 5 \end{cases}$$

1.6 $$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 &= -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + x_2 + 3x_3 + x_4 &= 11 \\ x_1 - 2x_2 + x_3 - 4x_4 &= -4 \end{cases}$$

2. Use Camer's rule to solve for z

$$\begin{cases} 4x + y + z + w &= 6 \\ 3x + 7y - z + w &= 1 \\ 7x + 3y - 5z + 8w &= -3 \\ x + y + z + 2w &= 3 \end{cases}$$

Chapter 4

Vector Spaces

4.1 Vector Space over Field

Definition 4.1.1: Field

A set F is a **field** defined by together with two operations, usually called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the following axioms hold

1. $\forall a, b \in F \quad a + b \in F \quad \text{and} \quad a \cdot b \in F$ (Closed)
2. $\forall a, b \in F \quad a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$ (Commutative)
3. $\forall a, b, c \in F \quad a + (b + c) = (a + b) + c$ (Associative for addition)
4. $\forall a, b, c \in F \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associative for multiplication)
5. $\exists 0 \in F \forall a \in F \quad a + 0 = a = 0 + a$ (Additive identity)
6. $\exists 1 \in F \forall a \in F \quad a \cdot 1 = a = 1 \cdot a$ (Multiplicative identity)
7. $\forall a \in F \quad a + (-a) = (-a) + a$ (Additive inverse)
8. $\forall a \in F - \{0\} \quad a \cdot a^{-1} = 1 = a^{-1} \cdot a$ (Multiplicative inverse)
9. $\forall a, b, c \in F \quad a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributive)

For example, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ where p is a prime. Then an element in a field is called **scalar**.

Definition 4.1.2: Vector Space over Field

Let V be a nonempty set of objects on which two perations are defined, addition and multiplication by scalars. If the following axioms are satisfied by all objects $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in V and all scalars k and ℓ in a field F, then we call V a **vector space** over F and we call objects in V **vectors**.

1. $\boldsymbol{u} + \boldsymbol{v} \in V$ (Closed)
2. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ (Commutative)
3. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$ (Associative)
4. $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u} = \mathbf{0} + \boldsymbol{u}$ for some $\mathbf{0} \in V$ (Identity)
5. $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0} = (-\boldsymbol{u}) + \boldsymbol{u}$ (Inverse)
6. $k\boldsymbol{u} \in V$ (Closed)
7. $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$
8. $(k + \ell)\boldsymbol{u} = k\boldsymbol{u} + \ell\boldsymbol{v}$
9. $k(\ell\boldsymbol{u}) = (k\ell)\boldsymbol{u}$
10. $1\boldsymbol{u} = \boldsymbol{u}$

Example 4.1.1 *Let $V = \mathbb{R}^2$ with the standard operations of addition and scalar multiplication defined by*

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2) \quad \textit{and} \quad k\boldsymbol{u} = (ku_1, ku_2)$$

where $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ belongs to $\mathbb{R}^2$ and a scalar $k \in \mathbb{R}$.
Then $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

In general, $n \in \mathbb{Z}$ with $n > 1$. We define the **standard operations** of addition and scalar multiplication to be

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_1 + v_2, ..., u_n + v_n) \quad \text{and} \quad k\boldsymbol{u} = (ku_1, ku_2, ..., ku_n)$$

where $\boldsymbol{u} = (u_1, u_2, ..., u_n)$, $\boldsymbol{v} = (v_1, v_2, ..., v_n)$, said to be **ordered $\boldsymbol{n}$-tuple**, belongs to $\mathbb{R}^n$, **$\boldsymbol{n}$-space**, and a scalar $k \in \mathbb{R}$. Then $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Example 4.1.2 *Let* $V = M_{22}(\mathbb{R})$ *where*

$$M_{22}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Show that $M_{22}(\mathbb{R})$ *is a vector space over* $\mathbb{R}$ *if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication.*

In general, $M_{mn}(\mathbb{R})$ is a vector space over $\mathbb{R}$ together with the operations of matrix addition and matrix scalar multiplication.

Example 4.1.3 *Let* $V = F(-\infty, \infty)$ *where*

$$F(-\infty, \infty) = \{f : f \text{ is a real-valued function on } (-\infty, \infty)\}.$$

Define

$$(\boldsymbol{f} + \boldsymbol{g})(x) = f(x) + g(x) \quad \text{and} \quad (k\boldsymbol{f})(x) = kf(x)$$

where $\boldsymbol{f}, \boldsymbol{g} \in F(-\infty, \infty)$ *and* $k \in \mathbb{R}$*. Show that* $F(-\infty, \infty)$ *is a vector space over* $\mathbb{R}$*.*

Example 4.1.4 *Let* $V = \{\mathbf{0}\}$ *and define*

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalar k*. Then* V *is a vector space called* ***zero vector space****.*

Example 4.1.5 *Let* $V = \mathbb{R}^2$ *and define*

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2) \quad \text{and} \quad k\boldsymbol{u} = (ku_1, 0)$$

where $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ *belongs to* $\mathbb{R}^2$ *and a scalar* $k \in \mathbb{R}$. *Show that* V *is not a vector space.*

Theorem 4.1.1

Let V is a vector space over a field F. Let $\boldsymbol{u} \in V$ and $k \in F$. Then

1. $0\boldsymbol{u} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $(-1)\boldsymbol{u} = -\boldsymbol{u}$
4. If $k\boldsymbol{u} = \mathbf{0}$, then $k = 0$ or $\boldsymbol{u} = \mathbf{0}$.

Exercise 4.1

1. Determine which sets are vector spaces under the given operations. For those that are not, list all axiom that fail to hold.

   1.1 Let $V = \mathbb{R}^2$ and define

   $$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \quad \text{and} \quad k(u_1, u_2) = (2ku_1, 2ku_2)$$

   1.2 Let $V = \mathbb{R}^2$ and define

   $$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 + 1, u_2 + v_2 + 1) \quad \text{and} \quad k(u_1, u_2) = (ku_1, ku_2)$$

   1.3 Let $V = \mathbb{R}^2$ and define

   $$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 v_2) \quad \text{and} \quad k(u_1, u_2) = (ku_1, ku_2)$$

   1.4 Let $V = \mathbb{R}^3$ and define

   $$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \text{and} \quad k(u_1, u_2, u_3) = (ku_1, u_2, u_3)$$

   1.5 Let $V = \mathbb{R}^3$ and define

   $$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \text{and} \quad k(u_1, u_2, u_3) = (0, 0, 0)$$

   1.6 Let $V = \{(x, x) : x \in \mathbb{R}\}$ and define the standard operations.

   1.7 Let $V = \{(x, 0) : x \in \mathbb{R}\}$ and define the standard operations.

   1.8 Let $V = \{(0, y) : y \in \mathbb{R}\}$ and define the standard operations.

   1.9 Let $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ with matrix addition and scalar multiplication.

   1.10 Let $V = \left\{ \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ with matrix addition and scalar multiplication.

   1.11 Let $V = \{a + bx : a, b \in \mathbb{R}\}$ and define

   $$(a_0 + a_1 x) + (b_0 + b_1 x) = (a_0 + b_0) + (a_1 + b_1)x \quad \text{and} \quad k(a_0 + a_1 x) = (ka_0) + (ka_1)x$$

   1.12 Let $V = \{f : \mathbb{R} \to \mathbb{R} : f(1) = 0\}$ and define in Example 4.1.3.

2. Prove that a line passing through the origin in $\mathbb{R}^3$ is a vector space under the standard operations on $\mathbb{R}^3$.

4.2 Subspaces

Definition 4.2.1: Subspace

A subset W of a vector space V over a field F is called a **subspace** of V, denoted $W \leq V$, if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem 4.2.1: Subspace Criterion

Let W be a nonempty subset of a vector space V over a field F. W is a subspace of V if and only if the following conditions hold.

1. $\forall \boldsymbol{u}, \boldsymbol{v} \in W \quad \boldsymbol{u} + \boldsymbol{v} \in W$
2. $\forall k \in F \forall \boldsymbol{u} \in W \quad k\boldsymbol{u} \in W$

Example 4.2.1 *Determine which of the following are subspaces of* $\mathbb{R}^2$.

1. $W = \{(x, x) : x \in \mathbb{R}\}$
2. $W = \{(x, x^2) : x \in \mathbb{R}\}$
3. $W = \{(x, 1) : x \in \mathbb{R}\}$
4. $W = \{(x, y) : x > 0 \text{ and } y > 0\}$

Example 4.2.2 *Show that* $W = \{(x, y, z) : x + y + z = 0\}$ *is a subspace of* $\mathbb{R}^3$.

Example 4.2.3 *Determine which of the following are subspaces of* $M_{nn}(F)$.

1. $W =$ the set of $n \times n$ symmetric matrices

2. $W =$ the set of $n \times n$ upper triangular matrices

3. $W =$ the set of $n \times n$ lower triangular matrices

4. $W =$ the set of $n \times n$ diagonal matrices

Example 4.2.4 *Determine which of the following are subspaces of* $M_{22}(\mathbb{R})$.

1. $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = c + b \right\}$

2. $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$

3. $W = \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

4. $W = \{A \in M_{22}(\mathbb{R}) : \det(A) \neq 0\}$

Example 4.2.5 *Show that the set of polynomial functions of degree* n

$$W = \mathbb{P}_n = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_i \in \mathbb{R}\}$$

is a subspace of $F(-\infty, \infty)$*.*

Example 4.2.6 *Show that the set of continuous functions*

$$W = C(-\infty, \infty) = \{f : \mathbb{R} \to \mathbb{R} : f \textit{ is continuous } \}$$

is a subspace of $F(-\infty, \infty)$*.*

Theorem 4.2.2

Let $A\boldsymbol{x} = \boldsymbol{0}$ be a homogeneous linear system of m equations in n unknowns. Define the set of solution vector

$$W = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{0}\}.$$

Then W is a subspace of $\mathbb{R}^n$. It is called the **solution space**.

Example 4.2.7 *Find the solution subspaces of the following linear systems.*

1. $$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & 8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4. $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition 4.2.2: Linear combination

A vector $\boldsymbol{w}$ is called a linear combination of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$, if it can be expressed in the form

$$\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

where $c_1, c_2, ..., c_k$ are scalars.

For example, every vector $(a, b, c) \in \mathbb{R}^3$ is expressible as a linear combination of $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ since

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}.$$

Example 4.2.8 *Let* $\boldsymbol{u} = (1, 2, -1)$ *and* $\boldsymbol{v} = (6, 4, 2)$*. Show that*

1. $\boldsymbol{w_1} = (9, 2, 7)$ is a linear combimnation of $\boldsymbol{u}$ and $\boldsymbol{v}$
2. $\boldsymbol{w_2} = (4, -1, 8)$ is not a linear combimnation of $\boldsymbol{u}$ and $\boldsymbol{v}$

Example 4.2.9 *Let* $\boldsymbol{p_1} = 1 + x^2$, $\boldsymbol{p_2} = 1 - x^2$ *and* $\boldsymbol{p_3} = 1 + x + x^2$. *Show that*

1. $3 - 5x + 4x^2$ is a linear combimnation of $\boldsymbol{p_1}$, $\boldsymbol{p_2}$ and $\boldsymbol{p_3}$

2. $3 + 7x - 3x^2$ is not a linear combimnation of $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$

Theorem 4.2.3

Let $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$ are vectors in a vector space V. Then

1. the set W of linear combinations of $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$ is a subspace of V

2. W is the smallest subspace of V that contain $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$ in the sense that every other subspace of V that caontains $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$ must contain W.

Definition 4.2.3: Spanning

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a set of vectors in a vector space V. Then the subspace W of V containing of all linear combination of the vectors in S is called **space spanned** by $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$, and we say that the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$ **span** W. We write

$$W = \text{span}(S) \quad \text{or} \quad W = \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$$

For example, $\mathbb{R}^2 = \text{span}\{\boldsymbol{i}, \boldsymbol{j}\}$ and $\mathbb{R}^3 = \text{span}\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$

Example 4.2.10 *Determine whether* $\boldsymbol{v_1} = (1, 1)$ *and* $\boldsymbol{v_2} = (-1, 2)$ *span the vector in space* $\mathbb{R}^2$.

Example 4.2.11 *Determine whether* $\boldsymbol{v_1} = (1, 1, 2)$, $\boldsymbol{v_2} = (1, 0, 1)$ *and* $\boldsymbol{v_2} = (2, 1, 3)$ *span the vector in space* $\mathbb{R}^3$.

Example 4.2.12 *Find a spanning set of $\mathbb{P}_n$.*

Theorem 4.2.4

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ and $S' = \{\boldsymbol{v'_1}, \boldsymbol{v'_2}, ..., \boldsymbol{v'_r}\}$ be two sets of vectors in a vector space V. Then

$$\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\} = \text{span}\{\boldsymbol{v'_1}, \boldsymbol{v'_2}, ..., \boldsymbol{v'_r}\}$$

if and only if each vector in S is a linear combination of other in S' and each vector in S' is a linear combination of those in S.

Exercise 4.2

1. Determine which of the following are subspaces of $\mathbb{R}^2$.

   1.1 $W = \{(x, 0) : x \in \mathbb{R}\}$

   1.2 $W = \{(x^2, x) : x \in \mathbb{R}\}$

   1.3 $W = \{(x, y) : x + y = 1 \in \mathbb{R}\}$

   1.4 $W = \{(x, y) : x > 0 \text{ and } y = 0\}$

2. Determine which of the following are subspaces of $\mathbb{R}^3$.

   2.1 $W = \{(x, x, x) : x \in \mathbb{R}\}$

   2.2 $W = \{(x, x, x^2) : x \in \mathbb{R}\}$

   2.3 $W = \{(x, y, z) : x + y + 2z = 0\}$

   2.4 $W = \{(x, y, z) : x + y = z\}$

3. Determine which of the following are subspaces of $M_{22}(\mathbb{R})$.

   3.1 $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

   3.2 $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

   3.3 $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad = cb \right\}$

   3.4 $W = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R} \right\}$

4. Determine which of the following are subspaces of $M_{nn}(F)$.

   4.1 $W = \{A \in M_{nn}(F) : A^T = -A\}$

   4.2 $W = \{A \in M_{nn}(F) : AB = -BA\}$ for a fixed $B \in M_{nn}(F)$

5. Determine which of the following are subspaces of $\mathbb{P}_3$.

   5.1 $W = \{a_1x + a_2x^2 + a_3x^3 : a_1, a_2, a_3 \in \mathbb{R}\}$

   5.2 $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$

   5.3 $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 + a_2 + a_3 = 0\}$

6. Determine which of the following are subspaces of $F(-\infty, \infty)$.

   6.1 $W = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is a constant function}\}$

   6.2 $W = \{f : \mathbb{R} \to \mathbb{R} : f(0) = 0\}$

   6.3 $W = \{f : \mathbb{R} \to \mathbb{R} : f(x) \leq 0 \text{ for all } x\}$

   6.4 $W = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable }\}$

7. Find the solution space of the system $A\boldsymbol{x} = \mathbf{0}$.

7.1 $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$ 7.3 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ 7.5 $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$

7.2 $A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & 9 \\ -2 & 4 & -6 \end{bmatrix}$ 7.4 $A = \begin{bmatrix} 1 & 2 & -6 \\ 1 & 4 & 4 \\ 3 & 10 & 6 \end{bmatrix}$ 7.6 $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$

8. Which of the following are linear combinations of $\boldsymbol{u} = (0, -2, 2)$ and $\boldsymbol{v} = (1, 3, -1)$?

8.1 $(2, 2, 2)$ 8.2 $(3, 1, 5)$ 8.3 $(0, 4, 5)$ 8.4 $(0, 0, 0)$

9. Express the following as linear combinations of $\boldsymbol{u} = (2, 1, 4)$, $\boldsymbol{v} = (1, -1, 3)$ and $\boldsymbol{w} = (3, 2, 5)$

9.1 $(-9, -7, -15)$ 9.2 $(6, 11, 6)$ 9.3 $(0, 0, 0)$ 9.4 $(7, 8, 9)$

10. Express the following as linear combinations of $\boldsymbol{p_1} = 2 + x + 4x^2$, $\boldsymbol{p_2} = 1 - x + 3x^2$ and $\boldsymbol{p_3} = 2 + 2x + 5x^2$

10.1 $9 - 7x - 15x^2$ 10.2 $6 + 11x + 6x^2$ 10.3 $7 + 8x + 9x^2$

11. In each part determine whether the given vectors span $\mathbb{R}^3$.

11.1 $\boldsymbol{v_1} = (2, 2, 2)$, $\boldsymbol{v_2} = (0, 0, 3)$, $\boldsymbol{v_3} = (0, 1, 1)$

11.2 $\boldsymbol{v_1} = (2, -1, 3)$, $\boldsymbol{v_2} = (4, 1, 2)$, $\boldsymbol{v_3} = (-8, 1, 8)$

11.3 $\boldsymbol{v_1} = (2, 2, 1)$, $\boldsymbol{v_2} = (2, 1, 2)$, $\boldsymbol{v_3} = (1, 2, 2)$

11.4 $\boldsymbol{v_1} = (1, 2, 3)$, $\boldsymbol{v_2} = (1, 0, 1)$, $\boldsymbol{v_3} = (0, 0, 1)$

12. Let $\boldsymbol{v_1} = (2, 1, 0, 3)$, $\boldsymbol{v_2} = (3, -1, 5, 2)$ and $\boldsymbol{v_3} = (-1, 0, 2, 1)$. Which of the following vectors are in span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.

12.1 $(2, 3, -7, 3)$ 12.3 $(1, 1, 1, 1)$

12.2 $(0, 0, 0, 0)$ 12.4 $(-4, 6, -13, 4)$

4.3 Linear Independence

Definition 4.3.1: Linear independence

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a set of vectors in a vector space. Then the vector equation

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{0}$$

has at least one solution, namely

$$c_1 = 0,\, c_2 = 0,\, ...,\, c_k = 0$$

If this is the only solution, then S is called a **linearly independent** set. If there are other solution, then S is called a **linearly dependent** set.

Example 4.3.1 *Determine which of the following are linearly independent.*

1. $S = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$
2. $S = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$
3. $S = \{(1, 1, 1), (2, 1, 1)\}$
4. $S = \{(1, 0), (0, 1), (1, 1)\}$
5. $S = \{(1, 2, 3)\}$
6. $S = \{(1, -1, 1), (-1, 2, 3), (0, 0, 0)\}$

Example 4.3.2 *Show that S is a linear dependent set.*

1. In $\mathbb{R}^4$, $\quad S = \{(2,-1,0,3),(1,2,5,-1),(7,-1,5,8)\}$

2. In $\mathbb{P}_2$, $\quad S = \{1-x, 5+3x-2x^2, 1+3x-x^2\}$

3. In $F(-\infty,\infty)$, $\quad S = \{\sin^2 x, \cos^2 x, 1\}$

Example 4.3.3 *Show that the polynomials* $1, x, x^2, ..., x^n$ *form a linearly independent set of vetors in* $\mathbb{P}_n$.

Theorem 4.3.1

Let S be a set of vectors with two or more vectors. S is linearly dependent if and only if at least one of vectors in S is expressible as a linear combination of the other vectors in S.

Theorem 4.3.2

1. A finite set of vectors that contains the zero vector is linearly dependent.

2. A set with exactly two vetors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem 4.3.3: Linear Dependent Test

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a set of vectors in $\mathbb{R}^n$. If $k > n$, then S is linearly dependent.

Next, we consider linearly independent of set of function $F(-\infty, \infty)$. Let

$$\boldsymbol{f_1} = f_2(x), \quad \boldsymbol{f_2} = f_2(x), ..., \quad \boldsymbol{f_n} = f_n(x)$$

be $n-1$ times differentiable functions in interval $(-\infty, \infty)$, the set of the functions denoted by $C^{n-1}(-\infty, \infty)$. Then the determinant of

$$W(x) = \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}$$

is called the **Wrinskian** of $\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}$.

Theorem 4.3.4: Linear Independent Test by Wronskian

Let $\boldsymbol{f_1}, \boldsymbol{f_2}, ..., \boldsymbol{f_n}$ be functions in $C^{n-1}(-\infty, \infty)$. If Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{n-1}(-\infty, \infty)$.

Example 4.3.4 *Show that*

1. $\boldsymbol{f_1} = x$ and $\boldsymbol{f_2} = \cos x$ form a linearly independent set of vectors in $C^1(-\infty, \infty)$.

2. $\boldsymbol{f_1} = 1$, $\boldsymbol{f_2} = e^x$ and $\boldsymbol{f_3} = e^{2x}$ form a linearly independent set of vectors in $C^2(-\infty, \infty)$.

Example 4.3.5 *Show that* $\boldsymbol{f_1} = 1$, $\boldsymbol{f_2} = \sin^2 x$ *and* $\boldsymbol{f_3} = \cos^2 x$ *form a linearly dependent set of vectors in* $C^2(-\infty, \infty)$.

Exercise 4.3

1. Explain why the following are linearly dependent sets of vectors.

   1.1 In $\mathbb{R}^3$, $S = \{(-1,1,4),(5,-10,-20)\}$

   1.2 In $\mathbb{R}^2$, $S = \{(3,-1),(4,5),(2,-7)\}$

   1.3 In $\mathbb{P}_2$, $S = \{3-2x+x^2, 6-4x+2x^2\}$

   1.4 In $F(-\infty,\infty)$, $S = \{\tan^2 x, \sec^2 x, 5\}$

   1.5 In $M_{22}(\mathbb{R})$, $S = \left\{ \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix} \right\}$

2. Which of the following sets of vectors in $\mathbb{R}^3$ are linearly dependent ?

   2.1 $(4,-1,2),(-4,10,2)$

   2.2 $(-3,0,4),(5,-1,2),(1,1,3)$

   2.3 $(8,-1,3),(4,1,0)$

   2.4 $(-2,0,1),(3,2,5),(6,-1,1),(1,1,3)$

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly dependent ?

   3.1 $(0,1,1,0),(1,1,0,0),(0,0,1,1)$

   3.2 $(3,8,7,-3),(1,5,3,-1),(2,-1,2,6),(1,4,0,3)$

   3.3 $(0,3,-3,0),(-2,0,0,0),(0,-4,-2,-2),(0,-8,4,-4)$

   3.4 $(2,0,1,7),(2,5,6,0),(1,9,9,9),(1,1,1,1)$

4. Which of the following sets of vectors in $\mathbb{P}_2$ are linearly dependent ?

   4.1 $2-x+4x^2, 3+6x+2x^2, 2+10x-4x^2$

   4.2 $6-x^2, 1+x+x^2$

   4.3 $3+x+x^2, 2-x+5x^2, 4-3x^2$

   4.4 $1+3x, x+4x^2, 5+6x-x^2, 7-2x+x^2$

5. Use the Wronskian to show that the following sets of vectors are linearly independent or dependent.

   5.1 $1, x, x^2$

   5.2 $1, x, e^x$

   5.3 $\sin x, \cos x, x$

   5.4 e^x, xe^x, x^2e^x

   5.5 $3\sin^2 x, 2\cos^2 x, 6$

   5.6 $\sin^2 x, \cos^2 x, x$

6. Show that if $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ is a linearly independent set of vectors, then $S = \{\boldsymbol{v_2}, \boldsymbol{v_3}, ..., \boldsymbol{v_k}\}$ is a linearly independent.

7. Show that if $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ is a linearly dependent set of vectors in a vector space V, then $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}, \boldsymbol{u}\}$ is a linearly independent for some $u \in V$.

4.4 Basis and Dimension

Definition 4.4.1: Basis

Let $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a set of vectors of a vector space. Then $\mathcal{B}$ is called a **basis** for V if the following two conditions hold.

1. $\mathcal{B}$ is linearly independent
2. $\mathcal{B}$ spans V

Example 4.4.1 *Show that*

1. $\mathcal{B} = \{\boldsymbol{i}, \boldsymbol{j}\}$ is a basis for $\mathbb{R}^2$
2. $\mathcal{B} = \{(1,1), (1,0)\}$ is a basis for $\mathbb{R}^2$

In general, setting

$$\boldsymbol{e_1} = (1,0,0,...,0), \quad \boldsymbol{e_2} = (0,1,0,...,0),..., \quad \boldsymbol{e_n} = (0,0,...,0,1)$$

which is called the **standard basis** for $\mathbb{R}^n$.

Example 4.4.2 *Show that* $\mathcal{B} = \{1, x, x^2\}$ *is a basis for* $\mathbb{P}_2$.

Theorem 4.4.1: Uniqueness of Basis Representation

Let $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be a basis for a vector space V. Then every vector $\boldsymbol{v}$ in V can be expressed in the form

$$\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

in exactly one way.

Coordinates Relative to a Basis. If $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ is a basis for a vector space V, and

$$\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

is the expression for a vector $\boldsymbol{v}$ in terms of the basis $\mathcal{B}$, then the scalar $c_1, c_2, ..., c_k$ are called the **coordinates** of $\boldsymbol{v}$ relative to the basis $\mathcal{B}$. The vector $(c_1, c_2, ..., c_k)$ in $\mathbb{R}^n$ constructed form these coordinates is called the **coordinate vector of $\boldsymbol{v}$ relative to $\mathcal{B}$**; it is denoted by

$$(\boldsymbol{v})_{\mathcal{B}} = (c_1, c_2, ..., c_k)$$

Example 4.4.3 *Let $\mathcal{B} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is a standard basis for $\mathbb{R}^3$ and $\boldsymbol{v} = (a, b, c) \in \mathbb{R}^3$. Find $(\boldsymbol{v})_{\mathcal{B}}$.*

Example 4.4.4 *Let $\mathcal{B} = \{1, x, x^2\}$ is a standard basis for $\mathbb{P}_2$ and $\boldsymbol{p} = a_0 + a_1x + a_2x^2$. Find $(\boldsymbol{p})_{\mathcal{B}}$. In general, $\{1, x, x^2, ..., x^n\}$ is a standard basis for $\mathbb{P}_n$.*

Example 4.4.5 *Let* $\boldsymbol{v} = (1, 2, 3)$.

1. Show that $\mathcal{B}_1 = \{(1, 1, 0), (0, 1, 1), (1, 0, 0)\}$ is a basis for $\mathbb{R}^3$
2. Show that $\mathcal{B}_2 = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ is a basis for $\mathbb{R}^3$
3. Compute $(\boldsymbol{v})_{\mathcal{B}_1}$ and $(\boldsymbol{v})_{\mathcal{B}_2}$

Example 4.4.6 *Let* $\boldsymbol{p} = 2 - 3x + 4x^2$.

1. Show that $\mathcal{B}_1 = \{1, 1 + x, x^2\}$ is a basis for $\mathbb{P}_2$
2. Compute $(\boldsymbol{p})_{\mathcal{B}}$

Example 4.4.7 *Let*

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1. Show that $\mathcal{B} = \{M_1, M_2, M_3, M_4\}$ is a standard basis for $M_{22}(\mathbb{R})$.
2. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find $(M)_{\mathcal{B}}$

Example 4.4.8 *Write the standard basis for $M_{23}(\mathbb{R})$.*

Definition 4.4.2: Finite-dimensional

A nonzero vector space V is called **finite-dimensional** if it contains a finite set of vectors

$$\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$$

that forms a basis. If no such set exists, V is called **infinite-dimensional**. In addtion, we shall regard the zero vector space to be finite-dimensional.

For example, the vector spaces $\mathbb{R}^n$, $\mathbb{P}_n$ and $M_{mn}(F)$ are finite-dimensional. The vector spaces $F(-\infty, \infty)$, $C(-\infty, \infty)$, $C^2(-\infty, \infty)$, $C^n(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$ are infinite-dimensional.

Theorem 4.4.2

Let V be a finite-dimensional vector space and $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ be any basis.

1. If a set has more than n vectors, then it is linearly dependent.
2. If a set has fewer than n vectors, then it does not span V.

Theorem 4.4.3

All bases for a finite-dimensional vector space have the same number of vectors.

Definition 4.4.3: Dimension

The **dimension** of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the zero vector space to have dimension zero.

For example, $\dim(\mathbb{R}^n) = n, \quad \dim(\mathbb{P}_n) = n + 1 \quad \text{and} \quad \dim(M_{mn}(F)) = mn.$

Example 4.4.9 *Determine a basis for and the dimension of the solution space of the homogeneous system*

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 & = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 & = 0 \\ x_1 + x_2 - x_3 - x_5 & = 0 \\ x_3 + x_4 + x_5 & = 0 \end{cases}$$

Theorem 4.4.4: Inserting and Removing a vector

Let S be a nonempty set of vectors in a vector space V.

1. If S is a linearly independent set, and if $\boldsymbol{v} \in V$ that is outside of $\text{span}(S)$, then the set $S \cup \{\boldsymbol{v}\}$ that results by inserting $\boldsymbol{v}$ into S is still linearly independent.

2. If $\boldsymbol{v} \in S$ that is expressible as a linear combination of other vectors in S, and if $S - \{\boldsymbol{v}\}$ denotes the set obtained by removing $\boldsymbol{v}$ from S, then S and $S - \{\boldsymbol{v}\}$ span the same space; that is
$$\text{span}(S) = \text{span}(S - \{\boldsymbol{v}\})$$

Theorem 4.4.5: Basis Test

If V is an $n-$dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.

Example 4.4.10 *Use Theorem 4.4.5 to show that*

1. $\mathcal{B} = \{(1,2),(3,4)\}$ is a basis for $\mathbb{R}^2$.

2. $\mathcal{B} = \{(2,0,-1),(4,0,7),(-1,1,4)\}$ is a basis for $\mathbb{R}^3$.

Theorem 4.4.6: Reducing and Inserting to construc a basis

Let S be a finite set of vectors in a finite-dimensional vector space V.

1. If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

2. If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Example 4.4.11 *Use Theorem 4.4.6 to find a basis from S for* $\mathbb{R}^3$

1. $S = \{(1,1,1),(1,2,1),(0,0,1),(1,0,1)\}$

2. $S = \{(2,0,1),(1,0,3)\}$

Exercise 4.4

1. Explan why the following sets of vectors are not bases for the indicate vector space.

    1.1 $\boldsymbol{v_1} = (1,2)$, $\boldsymbol{v_2} = (0,3)$, $\boldsymbol{v_3} = (2,5)$ for $\mathbb{R}^2$

    1.2 $\boldsymbol{v_1} = (-1,1,3)$, $\boldsymbol{v_2} = (6,1,1)$ for $\mathbb{R}^3$

    1.3 $\boldsymbol{p_1} = 1 + x + x^2$, $\boldsymbol{p_2} = 1 + x$ for $\mathbb{P}_2$

    1.4 $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}, D = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 7 & 1 \\ 9 & 2 \end{bmatrix}$ for $\mathbb{M}_{22}(\mathbb{R})$

2. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

    2.1 $(2,1),(1,2)$

    2.2 $(4,1),(-7,-8)$

    2.3 $(0,0),(1,3)$

    2.4 $(3,9),(4,12)$

3. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

    3.1 $(1,0,0),(2,2,0),(3,3,3)$

    3.2 $(3,1,-4),(2,5,6),(1,9,9)$

    3.3 $(2,-3,1),(4,1,1),(0,-2,7)$

    3.4 $(1,6,4),(2,4,-1),(-1,2,5)$

4. Which of the following sets of vectors are bases for $\mathbb{P}_2$?

    4.1 $1 - 3x + 2x^2, 1 + x + x^2, 1 - x$

    4.2 $4 + 6x + x^2, -1 + 4x + 2x^2, 5 + 2x - x^2$

    4.3 $1 + x + x^2, x + x^2, x^2$

    4.4 $-2 + x + 3x^2, 5 - 5x + x^2, 8 + 4x + x^2$

5. Show that the following sets of vectors is a basis for $M_{22}(\mathbb{R})$

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

6. Let V be the space spaned by $\boldsymbol{v_1} = \cos^2 x$, $\boldsymbol{v_2} = \sin^2 x$, $\boldsymbol{v_3} = \cos 2x$.

    6.1 Show that $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is not a basis for V.

    6.2 Find a basis for V.

7. Find $(\boldsymbol{w})_{\mathcal{B}}$ where $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is a basis for $\mathbb{R}^2$

   7.1 $\boldsymbol{v_1} = (1,0), \boldsymbol{v_2} = (1,1);\ \boldsymbol{w} = (3,-5)$

   7.2 $\boldsymbol{v_1} = (2,-4), \boldsymbol{v_2} = (3,5);\ \boldsymbol{w} = (1,7)$

   7.3 $\boldsymbol{v_1} = (1,3), \boldsymbol{v_2} = (3,1);\ \boldsymbol{w} = (9,2)$

   7.4 $\boldsymbol{v_1} = (1,5), \boldsymbol{v_2} = (-1,-7);\ \boldsymbol{w} = (a,b)$

8. Find $(\boldsymbol{w})_{\mathcal{B}}$ where $\mathcal{B} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is a basis for $\mathbb{R}^3$

   8.1 $\boldsymbol{v_1} = (3,3,3), \boldsymbol{v_2} = (1,0,0), \boldsymbol{v_3} = (2,2,0);\ \boldsymbol{w} = (1,2,3)$

   8.2 $\boldsymbol{v_1} = (1,2,3), \boldsymbol{v_2} = (-4,5,6), \boldsymbol{v_3} = (-7,8,9);\ \boldsymbol{w} = (5,-12,3)$

9. Find the coordinate vector of A relative to the basis $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ for $M_{22}(\mathbb{R})$

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}; \quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

10. Determine a basis for and the dimension of the solution space of the homogeneous system

   10.1 $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 + x_3 = 0 \end{cases}$

   10.2 $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

   10.3 $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

   10.4 $\begin{cases} 3x_1 - 4x_2 + 3x_3 - x_4 = 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 = 0 \end{cases}$

11. Let W be a subspace of a finite-dimensional vector space V. Prove that $\dim(W) \leq \dim(V)$.

12. Prove that if W is a subspace of a finite-dimensional vector space V and $\dim(W) = \dim(V)$, then $W = V$.

13. Let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be a basis for a vector space V. Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is also a basis, where

$$\boldsymbol{u_1} = \boldsymbol{v_1}, \quad \boldsymbol{u_2} = \boldsymbol{v_1} + \boldsymbol{v_2} \quad \text{and} \quad \boldsymbol{u_3} = \boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_3}$$

4.5 Row Space, Column Space and Nullspace

Definition 4.5.1: Row and Column vectors

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \boldsymbol{r_1} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \boldsymbol{r_2} &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\quad \vdots \\ \boldsymbol{r_m} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in $\mathbb{R}^n$ formed from the rows of A are called the **row vectors** of A, and the vector

$$\boldsymbol{c_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \boldsymbol{c_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots \quad \boldsymbol{c_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in $\mathbb{R}^m$ formed from the columns of A are called the **column vectors** of A.

Example 4.5.1 *Write row and column vectors of A.*

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 8 & 5 \\ 0 & 1 & 6 \\ 1 & 3 & -1 \end{bmatrix}$

Definition 4.5.2: Row space, Column space and Nullspace

Let A be an $m \times n$ matrix.

- The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A, denoted by Row$A$.
- The subspace of $\mathbb{R}^n$ spanned by the column vectors of A is called the **column space** of A, denoted by Col$A$.
- The solution space of the homogeous system of equations $A\boldsymbol{x} = \mathbf{0}$, which is a subspace of $\mathbb{R}^n$, is called the **nullspace** of A , denoted by Nul$A$; that is

$$\text{Nul}A = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \mathbf{0}\}$$

Example 4.5.2 *Find RowA, ColA and NulA.*

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

Theorem 4.5.1

A system of linear equations

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is consistent} \quad \text{if and only if} \quad \boldsymbol{b} \in \text{Col}A.$$

Example 4.5.3 *Let $A\boldsymbol{x} = \boldsymbol{b}$ be linear the system*

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\boldsymbol{b}$ is in ColA, and express $\boldsymbol{b}$ as a linear combination of the column vectors of A.

Theorem 4.5.2

Let $\boldsymbol{x_0}$ be any single solution of a consistent linear system $A\boldsymbol{x} = \boldsymbol{b}$ and $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}\}$ form a basis for the nulspace of A. Then every solution of $A\boldsymbol{x} = \boldsymbol{b}$ can be expressed in the form

$$\boldsymbol{x} = \boldsymbol{x_0} + c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

and, conversely, for all choices of scalar $c_1, c_2, ..., c_k$, the vector $\boldsymbol{x}$ in this formula is a solution of $A\boldsymbol{x} = \boldsymbol{b}$.

The vector $\boldsymbol{x_0}$ is called a **particular solution** of $A\boldsymbol{x} = \boldsymbol{b}$. The expression

$$\boldsymbol{x_0} + c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

is called the **general solution** of $A\boldsymbol{x} = \boldsymbol{b}$ and

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$$

is called the **general solution** of $A\boldsymbol{x} = \boldsymbol{0}$.

Example 4.5.4 *Find a particular solution and the general solution of linear system*

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 & = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 & = -1 \\ 5x_3 + 10x_4 + 15x_6 & = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 & = 6 \end{cases}$$

Theorem 4.5.3

1. EROs do not change the nullspace of a matrix.

2. EROs do not change the row space of a matrix.

Example 4.5.5 *Find a basis for the nullspace of*

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Theorem 4.5.4

Let A and B be row equivalent matrices.

1. A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

2. A given set of column vectors of A forms a basis for Col$A$ if and only if the corresponding column vectors of B forms a basis for Col$B$.

Theorem 4.5.5

If a matrix R is in REF, then the row vectors with the leading 1's form a basis for Row$R$, and the column vectors with leading 1's of the row vectors form a basis for Col$R$.

Example 4.5.6 *The matrix*

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in REF. Find bases for RowR and ColR.

Example 4.5.7 *Find bases for row and column spaces of*

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Example 4.5.8 *Find a basis for the space spanned by the vectors*

$$\boldsymbol{v_1} = (1, -2, 0, 0, 3),\ \boldsymbol{v_2} = (2, -5, -3, -2, 6),\ \boldsymbol{v_3} = (0, 5, 15, 10, 0),\ \boldsymbol{v_4} = (2, 6, 18, 8, 6).$$

Example 4.5.9 *Find a basis for the row space of*

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Example 4.5.10 *Find a subset of the vectors*

$$\boldsymbol{v_1} = (1, -2, 0, 3),\ \boldsymbol{v_2} = (2, -5, -3, 6),\ \boldsymbol{v_3} = (0, 1, 3, 0),\ \boldsymbol{v_4} = (2, -14, -7),\ \boldsymbol{v_5} = (5, -8, 1, 2)$$

that forms a basis for the space spanned by these vectors.

Exercise 4.5

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -2 \\ 1 & 4 & 2 & 3 \end{bmatrix}$$

2. Determine whether $\boldsymbol{b}$ is in the column space of A, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of A.

2.1 $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}$; $\boldsymbol{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

2.2 $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$; $\boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

2.3 $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $\boldsymbol{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

2.4 $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$; $\boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

3. Suppose that $x_1 = -1, x_2 = 2, x_3 = 4, x_4 = -3$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

3.1 Find the vectors form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

3.2 Find the vectors form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

4. Find the vectors form of the general solution of the given linear system $A\boldsymbol{x} = \boldsymbol{b}$; then use that result to find the vector form of the general solution of the $A\boldsymbol{x} = \boldsymbol{0}$.

4.1 $\begin{cases} x_1 - 3x_2 &= 1 \\ 2x_1 - 6x_2 &= 2 \end{cases}$

4.2 $\begin{cases} x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{cases}$

4.3 $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 &= -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 &= -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 &= 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 &= 3 \end{cases}$

4.4 $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 &= 4 \\ -2x_1 + x_2 + 2x_3 + x_4 &= -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 &= 3 \\ 4x_1 - 7x_2 - 5x_4 &= -5 \end{cases}$

5. Find a basis for the Nul$A$.

5.1 $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

5.2 $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

5.3 $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

5.4 $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

6. In each part, a matrix in REF form is given. By inspection, find bases for Row$A$ and Col$A$.

6.1 $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

6.2 $A = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

6.3 $A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

7. Find a basis for subspace of $\mathbb{R}^4$ spanned by given vectors.

7.1 $(1, 1, -4, -3), (2, 0, 2, -2), (2, -1, 3, 2)$

7.2 $(-1, 1, -2, 0), (3, 3, 6, 0), (9, 0, 0, 3)$

7.3 $(1, 1, 0, 0), (0, 0, 1, 1), (-2, 0, 2, 2), (0, -3, 0, 3)$

8. Find a suset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

8.1 $\boldsymbol{v_1} = (1, 0, 1, 1), \boldsymbol{v_2} = (-3, 3, 7, 1), \boldsymbol{v_3} = (-1, 3, 9, 3), \boldsymbol{v_4} = (-5, 3, 5, -1)$

8.2 $\boldsymbol{v_1} = (1, -2, 0, 3), \boldsymbol{v_2} = (2, -4, 0, 6), \boldsymbol{v_3} = (-1, 1, 2, 0), \boldsymbol{v_4} = (0, -1, 2, 3)$

8.3 $\boldsymbol{v_1} = (1, -1, 5, 2), \boldsymbol{v_2} = (-2, 3, 1, 0), \boldsymbol{v_3} = (4, -5, 9, 4), \boldsymbol{v_4} = (0, 4, 2, -3), \boldsymbol{v_5} = (-7, 18, 2, -8)$

4.6 Rank and Nullity

Theorem 4.6.1: Fundamental Matrix Space

The row space and column space of any mtrix have the same dimension.

Definition 4.6.1: Rank and Nullity

Let A be any matrix.

- The common dimension of Row$A$ and Col$A$ is called the **rank** of A, denoted by rank(A).
- The dimension of Nul$A$ is called the **nullity** of A, denoted by nullity(A).

Example 4.6.1 *Find the rank and nullity of the matrix and its transpose.*

$$\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Theorem 4.6.2: Rank of Transpose Matrix

Let A be any matrix. Then

$$\text{rank}(A) = \text{rank}(A^T)$$

Theorem 4.6.3: Dimension Theorem for Matrices

Let A be $m \times n$ matrix. Then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Note that $\text{rank}(A) \leq \min\{m, n\}$

Theorem 4.6.4

Let A be an $m \times n$ matrix. Then

1. $\text{rank}(A) =$ the number of leading variables in the solution of $A\boldsymbol{x} = \mathbf{0}$.

2. $\text{nullity}(A) =$ the number of parameter in the general solution of $A\boldsymbol{x} = \mathbf{0}$.

Example 4.6.2 *Find the rank and nullity of the matrix*

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

For any $m \times n$ matrix A, by theorem 4.6.2 and theorem 4.6.3, we obtain the following table.

Fundametal Space	**Dimension**
Row$A$	r
Col$A$	r
Nul$A$	$n-r$
NulA^T	$m-r$

Theorem 4.6.5: The consistency Theorem

Let $A\boldsymbol{x} = \boldsymbol{b}$ be a linear system of m equations in n unknowns. Then TFAE.

1. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

2. $\boldsymbol{b} \in \mathrm{Col}A$.

3. The coefficient matrix A and argumented matrix $[A \mid \boldsymbol{b}]$ have the same rank.

Example 4.6.3 *Consider rank(A) and rank$([A \mid \boldsymbol{b}])$ for the linear system $A\boldsymbol{x} = \boldsymbol{b}$.*

$$\begin{cases} x_1 - 2x_2 - 3x_3 + 2x_4 & = -4 \\ -3x_1 - 7x_2 - x_3 + x_4 & = -3 \\ 2x_1 - 5x_2 + 4x_3 - 3x_4 & = 7 \\ -3x_1 + 6x_2 + 9x_3 - 6x_4 & = -1 \end{cases}$$

Theorem 4.6.6

Let $A\boldsymbol{x} = \boldsymbol{b}$ be a linear system of m equations in n unknowns. Then TFAE.

1. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $m \times 1$ matrix $\boldsymbol{b}$.

2. The column vectors of A span $\mathbb{R}^m$.

3. $\operatorname{rank}(A) = m$.

A linear system with more equations than unknowns is called an **overdetermined linear system**.

Example 4.6.4 *Find condition b's in the overdetermined linear system.*

$$\begin{cases} x_1 - 2x_2 &= b_1 \\ x_1 - x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 + 2x_2 &= b_4 \\ x_1 + 3x_2 &= b_5 \end{cases}$$

Theorem 4.6.7

Let $A\boldsymbol{x} = \boldsymbol{b}$ be a consistent linear system of m equations in n unknowns. and $\operatorname{rank}(A) = r$. Then the general solution of the system contains $n - r$ parameters.

Theorem 4.6.8

Let A be any $m \times n$ matrix. Then TFAE.

1. $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

2. The column vectors of A are linearly independent.

3. $A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution (none or one) for every $m \times 1$ matrix $\boldsymbol{b}$.

Theorem 4.6.9: Equivalent Statements

Let A be an $n \times n$ matrix. Then TFAE.

1. A is invertible.
2. $A\boldsymbol{x} = \mathbf{0}$ has only trivial solution.
3. The RREF of A is I_n.
4. A is expressible as a product of elementary matrices.
5. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.
6. $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.
7. $\det(A) \neq 0$.
8. $\text{rank}(A) = n$.
9. $\text{nullity}(A) = 0$.
10. The row vectors of A are linearly independent.
11. The column vectors of A are linearly independent.
12. The row vectors of A span $\mathbb{R}^n$.
13. The column vectors of A span $\mathbb{R}^n$.
14. The row vectors of A form a basis for$\mathbb{R}^n$.
15. The column vectors of A form a basis for$\mathbb{R}^n$.

Exercise 4.6

1. Verify that rank(A) = rank(A^T).

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nulity of the matrix; then verify that the values obtained satisfy the dimension theorem.

2.1 $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

2.2 $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

2.3 $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

2.4 $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

2.5 $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. What conditions must satisfied by b_1, b_2, b_3, b_4 and b_5 for the overdetermined linear system

$$\begin{cases} x_1 - 3x_2 &= b_1 \\ x_1 - 2x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 - 4x_2 &= b_4 \\ x_1 + 5x_2 &= b_5 \end{cases}$$

4. Are there values of r and s for which the rank of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

is one or two ? If so, find those values.

Chapter 5

Linear Transformations

5.1 Linear Functions

Definition 5.1.1: Linerar Function

Let $T : V \to W$ be a function from a vector space V into a vector space W. Then T is called a **linear transformation** from V to W if for all vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ and all scalars c

(a) $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$

(b) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$

In special case, if $T : V \to V$ is a linear transformation, it is called a **linear operator**.

Example 5.1.1 *Let V and W be vector spaces and $T : V \to W$ be a function. Let k be any scalar. Determine whether T is a linear transformation if we define:*

1. $T(\boldsymbol{v}) = 0 \quad$ for all $\boldsymbol{v} \in V$
2. $T(\boldsymbol{v}) = kv \quad$ for all $\boldsymbol{v} \in V$

Example 5.1.2 *Determine whether T is a linear transformation.*

1. Let $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = x_1$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = x_1 + x_2$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (2x_1, 3x_2)$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (0, x_1 + x_2, x_1x_2)$

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + x_3)$

Example 5.1.3 *Let* $T : \mathbb{P}_n \to \mathbb{P}_{n+1}$ *and* $\boldsymbol{p} = p(x) = a_0 + a_1 x + \cdots + a_n x^n$. *Defnine*

$$T(\boldsymbol{p}) = T(p(x)) = xp(x)$$

Show that T *is a linear transformation.*

Example 5.1.4 *Let* $T : C(-\infty, \infty) \to C(-\infty, \infty)$. *Defnine*

$$T(\boldsymbol{f}) = \int_0^x f(x)\, dx$$

Show that T *is a linear operator.*

Example 5.1.5 *Let V and W be vector spaces and $T : V \to W$ be a function. If $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ be a basis for V. Define*

$$T(\boldsymbol{v}) = (\boldsymbol{v})_B \qquad \text{for all } \boldsymbol{v} \in V$$

Show that T is a linear transformation.

Example 5.1.6 *Let $A \in M_{nm}(\mathbb{R})$ be a matrix and $T_A : R^m \to \mathbb{R}^n$ be a function defined*

$$T_A(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \mathbb{R}^m$$

Show that T is a linear transformation.

Theorem 5.1.1

If $T : V \to W$ be a linear transformation, then

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\boldsymbol{v}) = -T(\boldsymbol{v}) \quad$ for all $\boldsymbol{v} \in V$
3. $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v}) \quad$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$

Finding linear transformations from images of basis vectors

Example 5.1.7 *Let $\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. Compute the formula of T.*

1. Let $B = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and

$$T(e_1) = (1, 1) \quad T(e_2) = (-1, 2) \quad T(e_3) = (3, 4)$$

2. Let $B = \{(1,1,1),(1,1,0),(1,0,0)\}$ be the standard basis for $\mathbb{R}^3$ and

$$T(1,1,1) = (1,0) \quad T(1,1,0) = (-1,2) \quad T(1,0,0) = (3,5)$$

Theorem 5.1.2

If $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations, then

$$(T_2 \circ T_1) : U \to W \text{ is also a linear transformation.}$$

Example 5.1.8 *Compute the formula of* $T_2 \circ T_1$

1. $T_1(x, y) = (x, y, x + y)$ and $T_2(x, y, z) = (y, -x, x + z)$

2. $T_1(x, y) = (2x + y, x + 2y)$ and $T_2(x, y) = (6x, y - 2x, x + 2y)$

3. $T_1(x, y, z) = (x + y, x + z, y + z)$ and $T_2(x, y, z) = (x + 2y, 2x, y + 2x, z + 2x)$

Exercise 5.1

1. Determine whether the function is a linear transformation. Justify your answer.

   1.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 - x_2)$

   1.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 - x_2 + x_3)$

   1.3 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1, x_2 + x_3, x_3)$

   1.4 Let $T : M_{mn}(\mathbb{R}) \to M_{nm}(\mathbb{R})$ defined by $T(A) = A^T$

   1.5 Let $T : M_{nn}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det(A)$

   1.6 Let $T : M_{22}(\mathbb{R}) \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$

   1.7 Let $T : F(-\infty, \infty) \to F(-\infty, \infty)$ defined by $T(f(x)) = f(x + 1)$

2. Let T be a linear transformation and B a basis for domain of T. Compute the formula of T.

   2.1 $T : \mathbb{R}^2 \to \mathbb{R}^2$, $B = \{(1, 1), (1, 2)\}$ and

   $$T(1, 1) = (-1, 1) \quad \text{and} \quad T(1, 2) = (2, 1)$$

   2.2 $T : \mathbb{R}^2 \to \mathbb{R}^3$, $B = \{(-2, 1), (1, 3)\}$ and

   $$T(-2, 1) = (-1, 1, 2) \quad \text{and} \quad T(1, 3) = (0, 3, 4)$$

   2.3 $T : \mathbb{R}^3 \to \mathbb{R}^2$, $B = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ and

   $$T(1, 2, 1) = (1, 0), \quad T(2, 9, 0) = (0, 2, 3) \quad \text{and} \quad T(3, 3, 4) = (-1, 2, 3)$$

3. Let $T_1(x_1, x_2) = (x_1+x_2, x_1, -x_2)$ and $T_2(x_1, x_2, x_3) = (x_1+x_2, x_2+x_3)$. Find $T_1 \circ T_2(x_1, x_2, x_3)$

5.2 Kernel and Range

Definition 5.2.1: Kernel

Let $T : V \to W$ be a linear transformation. Then

$$Ker(T) = \{\boldsymbol{v} \in V : T(\boldsymbol{v}) = \boldsymbol{0}\}$$

is called the **kernel** of T.

$$Ran(T) = \{\boldsymbol{w} \in W : T(\boldsymbol{v}) = \boldsymbol{w}, \boldsymbol{v} \in V\}$$

is called the **range** of T.

Example 5.2.1 *Let $T : R^2 \to \mathbb{R}^2$ be the linear operator given by the formula*

$$T(x, y) = (2x - y, -8x + 4y)$$

1. Which of the vectors $(5, 10), (3, 2)$ and $(1, 1)$ are in $Ker(T)$?

2. Which of the vectors $(1, -4), (5, 0)$ and $(-3, 12)$ are in $Ran(T)$?

Example 5.2.2 *Find $Ker(T)$ and $Ran(T)$.*

1. $T(x, y) = (x + y, x - y)$

2. $T(x, y) = (x, x + y, y)$

3. $T(x, y, z) = (x + y, 2x + 2y, z - x)$

Theorem 5.2.1

Let $T : V \to W$ be a linear transformation. Then

1. The kernel of T is a subspace of V.
2. The range of T is a subspace of W.

Definition 5.2.2: Rank and Nullity

Let $T : V \to W$ be a linear transformation.

- The dimension of range of T is called the **rank** of T, denoted by $\text{rank}(T)$
- The dimension of kernel of T is called the **nullity** of T, denoted by $\text{nullity}(T)$

Example 5.2.3 *Find rank(T) and nullity(T) in Example 5.2.2*

1. $T(x, y) = (x + y, x - y)$

2. $T(x, y) = (x, x + y, y)$

3. $T(x, y, z) = (x + y, 2x + 2y, z - x)$

Theorem 5.2.2

Let $A \in M_{nm}(\mathbb{R})$ be a matrix and $T_A : R^m \to \mathbb{R}^n$ be a function defined

$$T_A(\boldsymbol{v}) = A\boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \mathbb{R}^m$$

Then

1. $\text{nullity}(T_A) = \text{nullity}(A)$
2. $\text{rank}(T_A) = \text{rank}(A)$

Example 5.2.4 *Let $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be multiplication by*

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Find the rank and nullity of T_A.

Theorem 5.2.3: Dimension Theorem for Linear Transformation

Let $T : V \to W$ be a linear transformation. If V is a finite dimensional vector space, then

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

Exercise 5.2

1. Let $T : R^4 \to \mathbb{R}^3$ be the linear operator given by the formula

$$T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, 2x_1 + x_2 + x_3 - 4x_4, 6x_1 - 9x_3 + 9x_4)$$

   1.1 Which of the following vectors are in $Ker(T)$?

   (a) $(3, -8, 2, 0)$ (b) $(0, 0, 0, 1)$ (c) $(0, -4, 1, 0)$

   1.2 Which of the following vectors are in $Ran(T)$?

   (a) $(0, 0, 6)$ (b) $(1, 3, 0)$ (c) $(2, 4, 1)$

2. Find $Ker(T)$, $Ran(T)$, rank(T) and nullity(T)

   2.1 $T(x, y) = (x + 3y, x - 3y)$

   2.2 $T(x, y, z) = (x - y + 3z, 5x + 6y - 4z, 7x + 4y + 2z)$

   2.3 $T(x_1, x_2, x_3) = (2x_1 - x_3, 4x_1 - 2x_3, -2x_1 + x_3)$

3. Let $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ be multiplication by

$$A = \begin{bmatrix} 1 & 4 & 5 & 0 & 9 \\ 3 & -2 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 3 & 5 & 1 & 8 \end{bmatrix}$$

   Find the rank and nullity of T_A.

5.3 Inverse Linear Transformation

Definition 5.3.1: One-to-one

A linear transformation $T : V \to W$ is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W, i.e.,

$$\forall \boldsymbol{u}, \boldsymbol{v} \in V \quad T(\boldsymbol{u}) = T(\boldsymbol{v}) \quad \to \quad \boldsymbol{u} = \boldsymbol{v}$$

Example 5.3.1 *Determine whether the following linear transformation T is one-to-one.*

1. $T(x, y) = (x + y, x)$

2. $T(x, y, z) = (x + y + z, x + y, y + z)$

3. $T(x, y, z) = (2x + y - z, x - y, 2x - 2y)$

Example 5.3.2 *Let A be an $n \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is multiplication by A. Prove that*

T_A is one-to-one if and only if A is invertible.

Example 5.3.3 *Let* $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ *be multiplication by*

$$A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{bmatrix}$$

Determine whether T_A *is one-to-one.*

Theorem 5.3.1: Equivalent Statements

Let $T : V \to W$ be a linear transformation. Then TFAE

1. T is one-to-one
2. $Ker(T) = \{\mathbf{0}\}$
3. nullity$(T) = 0$

Theorem 5.3.2: Equivalent Statements

Let $T : V \to W$ be a linear transformation. If V is a finite-vector space. Then TFAE

1. T is one-to-one
2. $Ker(T) = \{\mathbf{0}\}$
3. nullity$(T) = 0$
4. $Ran(T) = V$

Definition 5.3.2: Inverse Transformation

Let a linear transformation $T : V \to W$ is one-to-one. Then $T^{-1} : Ran(T) \to V$ is called the **inverse** of T which maps vectors in W back into vectors in V.

$$T(\boldsymbol{v}) = \boldsymbol{w} \quad \leftrightarrow \quad \boldsymbol{v} = T^{-1}(\boldsymbol{w})$$

Example 5.3.4 *Find formula of inverse of T.*

1. $T(x, y) = (x + y, x - y)$

2. $T(x, y, z) = (3x + y, -2x - 4y + 3z, 5x + 4y - 2z)$

3. $T(x, y, z) = (x + 2y + z, x - y + z, x + y, -x - 2y - z)$

Example 5.3.5 *Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be multiplication by A. Determine whether T has an inverse; if ao, find a formula of $(T_A)^{-1}$*

1. $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}$

Example 5.3.6 *Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix}$. Determine whether T has an inverse; if so, find a formula of $(T_A)^{-1}$*

Theorem 5.3.3

Let $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations. Then

1. $T_2 \circ T_1$ is one-to-one
2. $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

Example 5.3.7 *Let $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operators given by the formula*

$$T_1(x, y) = (x + y, x - y) \quad \text{and} \quad T_2(x, y) = (2x + y, x - 2y)$$

1. *Show that T_1 and T_2 are one-to-one*
2. *Find formulas for $T_1^{-1}(x, y)$ and $T_2^{-1}(x, y)$ and $(T_2 \circ T_1)^{-1}(x, y)$*
3. *Verify that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$*

Exercise 5.3

1. Find $Ker(T)$ and determine whether the following linear transformation T is one-to-one.

   1.1 $T(x,y) = (y,x)$

   1.2 $T(x,y) = (x+y, y-x)$

   1.3 $T(x,y) = (2x+4y, 0)$

   1.4 $T(x,y) = (x-y, y-x, 2x-2y)$

   1.5 $T(x,y) = (y, x, x+y)$

   1.6 $T(x,y,z) = (x+y+z, x-y-z)$

2. Determine whether $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(p(x)) = p(x+1)$ is one-to-one.

3. Let $T : \mathbb{P}_n \to \mathbb{P}_{n+1}$ be the linear transformation

$$T(p(x)) = xp(x)$$

   Show that T is one-to-one.

4. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A. Determine whether T has an inverse; if so, find a formula of $(T_A)^{-1}$

   4.1 $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ 4.2 $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ 4.3 $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ 4.4 $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ -2 & 1 & 3 \end{bmatrix}$

5. In each part determine whether the linear operator $T : M_{22}(\mathbb{R}) \to M_{22}(\mathbb{R})$ is one-to-one. If so, find a formula of T^{-1}

   5.1 $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

   5.2 $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

   5.3 $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

5.4 Matrix and Linear Transformation

Let V be an $n-$dimensional vector space and W be an $m-$dimensional vector space. If we choose bases B and B' for V and W, respectively. Let $\boldsymbol{x} \in V$. The coordinate matrix $[\boldsymbol{x}]_B$ or $(\boldsymbol{x})_B$ will be a vector in $\mathbb{R}^n$. Let A be the standard matrix for this this transformation $T : V \to W$. Then

$$A[\boldsymbol{x}]_B = [T(\boldsymbol{x})]_{B'}$$

Then matrix A is called the matrix for T with respect to the bases B and B'.

Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ and $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\}$. Then

$$A = [[T(\boldsymbol{u}_1)]_{B'}|[T(\boldsymbol{u}_2)]_{B'}| \cdots [T(\boldsymbol{u}_n)]_{B'}]$$

denoted by $[T]_{B',B}$ and

$$[T]_{B',B}[\boldsymbol{x}]_B = [T(\boldsymbol{x})]_{B'}$$

If linear operator $T : V \to V$, then

$$[T]_B = [[T(\boldsymbol{u}_1)]_{B'}|[T(\boldsymbol{u}_2)]_{B'}| \cdots [T(\boldsymbol{u}_n)]_{B'}]$$

and

$$[T]_B[\boldsymbol{x}]_B = [T(\boldsymbol{x})]_B$$

Example 5.4.1 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for $\mathbb{R}^2$ or $[T]_B$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 5.4.2 *Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for $\mathbb{R}^2$ and $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ or $[T]_{B',B}$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}; \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Example 5.4.3 *Let* $T : \mathbb{P}_1 \to \mathbb{P}_2$ *be the linear transformation defined by*

$$T(p(x)) = xp(x)$$

Find $[T]_{B',B}$ *if* $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ *and* $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ *where*

$$\boldsymbol{u}_1 = 1, \boldsymbol{u}_2 = x \quad \textit{and} \quad \boldsymbol{v}_1 = 1, \boldsymbol{v}_2 = x, \boldsymbol{v}_3 = x^2$$

Example 5.4.4 *Let* $T : \mathbb{P}_2 \to \mathbb{P}_2$ *be the linear operator defined by*

$$T(p(x)) = p(2x + 1)$$

1. Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$
2. Use the indirectly procedure to compute $T(1 + 2x + 3x^2)$

Theorem 5.4.1

Let $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations. If B, B' and B'' are bases for U, V and W, respectively, then

$$[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''}[T_1]_{B'',B}$$

Example 5.4.5 *Let $T_1 : \mathbb{P}_1 \to \mathbb{P}_2$ and $T_2 : \mathbb{P}_2 \to \mathbb{P}_2$ be linear transformations defined by*

$$T_1(p(x)) = xp(x) \quad \text{and} \quad T_2(p(x)) = p(2x+1)$$

Let $B = \{1, x\}$ and $B' = \{1, x, x^2\}$ be bases for $\mathbb{P}_1$ and $\mathbb{P}_2$, respecrtively. Compute $[T_2 \circ T_1]_{B',B}$

Example 5.4.6 *Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear operator defined by*

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{bmatrix} \quad \text{and} \quad T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

Find $[T_2 \circ T_1]_{B',B}$ where $B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Theorem 5.4.2

Let $T : V \to V$ be a linear operator. If B and is a basis for V, then TFAE

1. T is one-to-one

2. $[T]_B$ is invertible

Moreover, when these equivalent condition hold

$$[T^{-1}]_B = [T]_B^{-1}$$

Example 5.4.7 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

Find $[T^{-1}]_B$ where $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

Exercise 5.4

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

and let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ be the basis for which

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1.1 Find $[T]_B$

1.2 Verify that fomula 1.1 hold for every vector $\boldsymbol{x}$ in $\mathbb{R}^2$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 3x_2 \\ x_1 + 5x_2 \end{bmatrix}$$

2.1 Find $[T]_{B',B}$ with respect to the bases $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ for $\mathbb{R}^2$ and $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

2.2 Verify that fomula 2.1 hold for every vector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$

3. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation defined by

$$T(p(x)) = xp(x)$$

Find $[T]_{B',B}$ if $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.

4. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear operator defined by

$$T(p(x)) = p(1 - 3x)$$

4.1 Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$

4.2 Use the indirectly procedure to compute $T(1-3x+2x^2)$

5. Let $T_1 : \mathbb{P}_2 \to \mathbb{P}_2$ and $T_2 : \mathbb{P}_2 \to \mathbb{P}_3$ be linear transformations defined by

$$T_1(p(x)) = p(5x+1) \quad \text{and} \quad T_2(p(x)) = xp(x+1)$$

Let $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$ be bases for $\mathbb{P}_2$ and $\mathbb{P}_3$, respecrtively. Compute $[T_2 \circ T_1]_{B',B}$

6. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear operator defined by

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 3x_1 + x_2 \end{bmatrix} \quad \text{and} \quad T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

Find $[T_2 \circ T_1]_{B',B}$ where $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ and $B' = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

Find $[T^{-1}]_B$ where $B = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$

Chapter 6

Inner Product

6.1 Inner Product

Definition 6.1.1: Inner Product

An **inner product** on a real vector space V is a function that associatives a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V in such a way that the following axioms are satisfied for all vector $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ in V and all scalars k.

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
3. $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k \langle \boldsymbol{u}, \boldsymbol{v} \rangle$
4. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$

A real space with an inner product is called a **real inner product space**.

Definition 6.1.2: Euclidean inner product

Let $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ and $\boldsymbol{v} = (v_1, v_2, ..., v_n)$ be elements in $\mathbb{R}^n$. The **Euclidean inner product** on $\mathbb{R}^n$ is defined by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_2$$

Example 6.1.1 *Compute the Euclidean inner products of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$

1. $\boldsymbol{u} = (1, 2, 3)$ and $\boldsymbol{v} = (-2, 1, 4)$
2. $\boldsymbol{u} = (2, -2, -2, 0)$ and $\boldsymbol{v} = (0, -3, 3, 5)$

Example 6.1.2 *Let* $\boldsymbol{u} = (u_1, u_2)$ *and* $\boldsymbol{v} = (v_1, v_2)$ *be vectors in* $\mathbb{R}^2$. *Verify that*

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2$$

satisfies the four inner product axiom

Example 6.1.3 *Let* $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ *be two matrices in* $M_{22}(\mathbb{R})$. *Show that*

$$\langle A, B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

is an inner product on $M_{22}(\mathbb{R})$.

Example 6.1.4 *Let $\boldsymbol{p} = a_0 + a_1x + a_2x^2$ and $\boldsymbol{q} = b_0 + b_1x + b_2x^2$ be vectors in $\mathbb{P}_2$. Verify that*

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

satisfies the four inner product axiom

Example 6.1.5 *Let $\boldsymbol{f} = f(x)$ and $\boldsymbol{g} = g(x)$ be two continuous functions in $C[a, b]$. Show that*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_a^b f(x)g(x)\, dx$$

is an inner product on $C[a, b]$.

Definition 6.1.3: Norm

Let V be an inner product space. The **norm** of a vector $\boldsymbol{u}$ in V is denoted by $\|\boldsymbol{u}\|$ and is defined by

$$\|\boldsymbol{u}\| = \langle \boldsymbol{u}, \boldsymbol{u} \rangle^{\frac{1}{2}}$$

In the Euclidean vector space with the Euclidean inner product, the norm (Euclidean norm) is

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Example 6.1.6 *Compute the Euclidean norm of the following verctors*

1. $\boldsymbol{u} = (3, 4)$
2. $\boldsymbol{u} = (2, 2, -1)$

Example 6.1.7 *Let* $\boldsymbol{u} = (u_1, u_2)$ *and* $\boldsymbol{v} = (v_1, v_2)$ *and define* $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2$ *Compute the norm of the following verctors*

1. $\boldsymbol{u} = (3, 4)$
2. $\boldsymbol{u} = (2, 2, -1)$

Example 6.1.8 *Compute* $\|U\| + \|V\|$ *and* $\|U + V\|$ *where* $U = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ *and* $V = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$

Example 6.1.9 *Compute* $\|\boldsymbol{f}\| + \|\boldsymbol{g}\|$ *and* $\|\boldsymbol{f} + \boldsymbol{g}\|$ *where* $\boldsymbol{f} = \cos x$ *and* $\boldsymbol{g} = x$ *on* $C[0, \pi]$

Theorem 6.1.1: Properties if inner product

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in a real inner product space and k be any scalar. Then

1. $\langle \mathbf{0}, \boldsymbol{u} \rangle = \langle \boldsymbol{u}, \mathbf{0} \rangle = 0$

2. $\langle \boldsymbol{u}, k\boldsymbol{v} \rangle = k \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$

3. $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

4. $\langle \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

5. $\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle - \langle \boldsymbol{v}, \boldsymbol{w} \rangle$

Example 6.1.10 *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in a real inner product space. Show that*

1. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle = \|\boldsymbol{u}\|^2 - \|\boldsymbol{v}\|^2$

2. $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2 \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2$

3. $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 - 2 \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \|\boldsymbol{v}\|^2$

4. $\langle \boldsymbol{u} - 2\boldsymbol{v}, 3\boldsymbol{u} + 4\boldsymbol{v} \rangle = 3\|\boldsymbol{u}\|^2 - 2 \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 8\|\boldsymbol{v}\|^2$

Example 6.1.11 *Suppose that* $\boldsymbol{u}, \boldsymbol{v}$ *and* $\boldsymbol{w}$ *be vectors such that*

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 1 \quad \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -3 \quad \langle \boldsymbol{u}, \boldsymbol{w} \rangle = 2 \quad \|\boldsymbol{u}\| = 5 \quad \|\boldsymbol{v}\| = 3 \quad \|\boldsymbol{w}\| = 7$$

Evaluate the given expression.

1. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle$
2. $\langle 3\boldsymbol{u} - \boldsymbol{v}, 2\boldsymbol{w} + \boldsymbol{v} \rangle$
3. $\langle 4\boldsymbol{w} - 3\boldsymbol{v}, \boldsymbol{u} + 2\boldsymbol{v} \rangle$
4. $\|\boldsymbol{u} - \boldsymbol{v}\|$
5. $\|5\boldsymbol{u} + 3\boldsymbol{w}\|$
6. $\|3\boldsymbol{u} - 2\boldsymbol{w}\|$

Exercise 6.1

1. Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Define

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 5u_1v_1 + 7u_2v_2$$

   1.1 Prove that $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ is an inner product.

   1.2 Compute $\langle (-1,3), (3,-1) \rangle$ and $\|(2,5)\|$

2. Let $\boldsymbol{p} = a_0 + a_1x + a_2x^2$ and $\boldsymbol{q} = b_0 + b_1x + b_2x^2$ be vectors in $\mathbb{P}_2$. Define

$$\langle \boldsymbol{p}, \boldsymbol{q} \rangle = a_0b_0 + 2a_1b_1 + 3a_2b_2$$

   2.1 Prove that $\langle \boldsymbol{p}, \boldsymbol{q} \rangle$ is an inner product.

   2.2 Compute $\langle 1 + x, 1 + x^2 \rangle$ and $\|1 + 2x + 2x^2\|$

3. Suppose that $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors such that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2 \quad \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -3 \quad \langle \boldsymbol{u}, \boldsymbol{w} \rangle = 5 \quad \|\boldsymbol{u}\| = 1 \quad \|\boldsymbol{v}\| = 2 \quad \|\boldsymbol{w}\| = 7$$

   Evaluate the given expression.

   3.1 $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} + \boldsymbol{v} \rangle$

   3.2 $\langle 2\boldsymbol{u} - \boldsymbol{v}, 2\boldsymbol{w} + 4\boldsymbol{v} \rangle$

   3.3 $\langle 3\boldsymbol{w} - 4\boldsymbol{v}, \boldsymbol{u} + 5\boldsymbol{v} \rangle$

   3.4 $\|2\boldsymbol{u} - 3\boldsymbol{v}\|$

4. Let $\boldsymbol{f} = f(x)$ and $\boldsymbol{g} = g(x)$ be two continuous functions in $C[0,1]$. Define an inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^1 f(x)g(x)\,dx$$

   is an inner product on $C[0,1]$. Let $\boldsymbol{f} = f(x) = \sin x$ and $\boldsymbol{g} = g(x) = x$. Compute

   4.1 $\langle \boldsymbol{f}, \boldsymbol{g} \rangle$

   4.2 $\langle \boldsymbol{f} + \boldsymbol{g}, \boldsymbol{f} - \boldsymbol{g} \rangle$

   4.3 $\|\boldsymbol{f} + \boldsymbol{g}\|$

   4.4 $\|2\boldsymbol{f} - \boldsymbol{g}\|$

6.2 Orthogonality

Theorem 6.2.1: Cauchy-Schwarz inequality

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in a real inner product space. Then

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, then we obtain

$$-1 \leq \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|} \leq 1$$

We define θ to be the **angle between $\boldsymbol{u}$ and $\boldsymbol{v}$**

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|} \quad \text{and} \quad 0 \leq \theta \leq \pi$$

Example 6.2.1 *Find the cosine of the angle θ between the vectors*

$$\boldsymbol{u} = (1, 2, 3, 0) \quad \textit{and} \quad \boldsymbol{v} = (0, 1, -2, 3)$$

Definition 6.2.1: Orthogonal

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in an inner product space are called **orthogonal** if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$.

Example 6.2.2 *Determine whether* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are orthogonal.*

1. $\boldsymbol{u} = (1, 1, 3, -2)$ and $\boldsymbol{v} = (2, 1, 1, -3)$ on $\mathbb{R}^4$

2. $\boldsymbol{u} = (2, 1, 3)$ and $\boldsymbol{v} = (1, -1, 2)$ on $\mathbb{R}^3$

3. $u = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$ on $M_{22}(\mathbb{R})$

4. $\boldsymbol{f} = x$ and $\boldsymbol{g} = x^2$ on $C[-1, 1]$

Theorem 6.2.2: Generlized Theorem of Pythagoras

et $\boldsymbol{u}$ and $\boldsymbol{v}$ be orthogonal vectors in an inner product space. Then

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

Example 6.2.3 *Let* $\boldsymbol{u} = (1, 0, 3, -2)$ *and* $\boldsymbol{v} = (0, 2, -2, 3)$ *on* $C[-1, 1]$. *Compute* $\|\boldsymbol{u}+\boldsymbol{v}\|^2$

Example 6.2.4 *Let* $\boldsymbol{f} = x$ *and* $\boldsymbol{g} = x^2$ *on* $C[-1, 1]$. *Compute* $\|\boldsymbol{f}+\boldsymbol{g}\|^2$

Example 6.2.5 *Let* $\boldsymbol{p} = 1 + x + x^2$ *and* $\boldsymbol{q} = 1 - 2x + x^2$ *on* $\mathbb{P}_2$. *Compute* $\|\boldsymbol{p}+\boldsymbol{q}\|^2$

Definition 6.2.2: Orthogonal Complements

Let W be a subspace of an inner product space V. Let $\boldsymbol{u}$ be a vector in V

1. $\boldsymbol{u}$ is said to be **orthogonal to W** if it is orthogonal to every vector in W.

2. The set of all vectors in V that are orthogonal to W is called the **orthogonal complement of W**

$$W^{\perp} = \{\boldsymbol{u} \in V : \langle \boldsymbol{u}, \boldsymbol{w} \rangle = 0, \text{ for all } \boldsymbol{w} \in W\}$$

Example 6.2.6 *Determine whether $\boldsymbol{u}$ is orthogonal to W and find $W^{\perp}$.*

1. $\boldsymbol{u} = (-1, 2)$ and $W = Span\{(2, 1)\}$

2. $\boldsymbol{u} = (-1, 0, 1)$ and $W = Span\{(1, 0, 1), (2, 3, -2)\}$

3. $\boldsymbol{u} = (-1, 3, 4, 1)$ and $W = Span\{(3, 1, 0, 0), (0, 0, -1, 4), (0, 1, 0, 1)\}$

Theorem 6.2.3: Prperties of Orthogonal Complements

Let W be a subspace of a finite-dimentional inner product space V. Then

1. $W^\perp \leq V$
2. $W \cap W^\perp = \{\mathbf{0}\}$
3. $(W^\perp)^\perp = W$

Theorem 6.2.4

Let A be an $m \times n$ matrix.

1. The nullspace of A and the row space of A are orthogonal complement in $\mathbb{R}^n$ with respect to the Euclidean inner product.
2. The nullspace of A^T and the column space of A are orthogonal complement in $\mathbb{R}^m$ with respect to the Euclidean inner product.

Example 6.2.7 *Let*

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Verify that the nullspace of A and the row space of A are orthogonal complement in $\mathbb{R}^5$ and the column space of A are orthogonal complement in $\mathbb{R}^4$

Exercise 6.2

1. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

   1.1 $\boldsymbol{u} = (-1, 3, 2)$ and $\boldsymbol{v} = (4, 2, -1)$

   1.2 $\boldsymbol{u} = (-1, 2, 1)$ and $\boldsymbol{v} = (1, 2, 1)$

   1.3 $\boldsymbol{u} = (-4, 6, -10, 1)$ and $\boldsymbol{v} = (2, 1, -2, 9)$

   1.4 $\boldsymbol{u} = (a, b, c)$ and $\boldsymbol{v} = (b, -a, 0)$

2. Compute cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

   2.1 $\boldsymbol{u} = (-1, 2, 2)$ and $\boldsymbol{v} = (2, 2, -1)$

   2.2 $\boldsymbol{u} = (0, 4, 3)$ and $\boldsymbol{v} = (1, -1, 1)$

   2.3 $\boldsymbol{u} = (-2, 3, 1, 0)$ and $\boldsymbol{v} = (2, -3, -1, 0)$

   2.4 $\boldsymbol{u} = (a, 0, a)$ and $\boldsymbol{v} = (0, a, a)$

3. Compute cosine of the angle between A and B.

   3.1 $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

   3.2 $A = \begin{bmatrix} 6 & -2 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 5 & 0 \end{bmatrix}$

4. Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal ?

   4.1 $\boldsymbol{u} = (2, 3, 1)$ and $\boldsymbol{v} = (4, k, 2)$

   4.2 $\boldsymbol{u} = (k, k, 1)$ and $\boldsymbol{v} = (k, 5, 6)$

5. Determine whether $\boldsymbol{u}$ is orthogonal to W and find $W^{\perp}$.

   5.1 $\boldsymbol{u} = (-2, 1)$ and $W = Span\{(1, 2), (2, 1)\}$

   5.2 $\boldsymbol{u} = (1, 0, 1)$ and $W = Span\{(0, 3, 0), (-4, 3, 4)\}$

6. What is the vector that is orthogonal to $(1, 2, -1)$ and $(3, 4, 0)$?

7. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x - 2y - 3z = 0\}$. Find $W^{\perp}$.

6.3 Orthonormal Bases

Definition 6.3.1: Orthogonal set

A set of vector in an inner product space is called an **orthogonal set** if all pair distinct vectors in the set are orthogonal. An orthogonal set in which each vector has unit norm is called **orthonormal**.

Example 6.3.1 *Determine whether the sets are orthogonal set or orthonormal.*

1. $\{(1,0),(0,-3)\}$

2. $\{(1,0,0),(0,1,0),(0,0,-1)\}$

3. $\{(0,1,0),(1,0,1),(1,0,-1)\}$

4. $\{(1,2,0),(-2,1,0),(0,1,1)\}$

5. $\{e_1,e_2,e_3,...,e_n\}$

Theorem 6.3.1

Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ be an orthonormal basis for an inner product space V and $\boldsymbol{u}$ be any vector in V. Then

$$\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 + \cdots + \langle \boldsymbol{u}, \boldsymbol{v}_n \rangle \boldsymbol{v}_n$$

or

$$[\boldsymbol{u}]_B = (\langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle, ..., \langle \boldsymbol{u}, \boldsymbol{v}_n \rangle)$$

Example 6.3.2 *Let* $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, *where*

$$\boldsymbol{v}_1 = (0, 1, 0), \quad \boldsymbol{v}_2 = (-\frac{4}{5}, 0, \frac{3}{5}), \quad \boldsymbol{v}_3 = (\frac{3}{5}, 0, \frac{4}{5})$$

be an orthonormal basis for $\mathbb{R}^3$. *Compute* $[(1, 2, 3)]_B$ *and* $[(-3, 2, 1)]_B$

Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ be an orthogonal basis for an inner product space V. Then normalizing each these vectors yeilds the orthonormal basis

$$B' = \left\{ \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}, \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}, ..., \frac{\boldsymbol{v}_n}{\|\boldsymbol{v}_n\|} \right\}$$

If $\boldsymbol{u}$ is a vector in V, then

$$\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2} \boldsymbol{v}_2 + \cdots + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_n \rangle}{\|\boldsymbol{v}_n\|^2} \boldsymbol{v}_n$$

Example 6.3.3 *Let* $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, *where*

$$\boldsymbol{v}_1 = (0, 5, 0), \quad \boldsymbol{v}_2 = (-3, 0, 4), \quad \boldsymbol{v}_3 = (4, 0, 3)$$

be an orthogonal basis for $\mathbb{R}^3$. *Compute* $[(1, 2, 3)]_B$ *and* $[(-3, 2, 1)]_B$

Theorem 6.3.2

If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ be an orthogonal set of nonzero vectorss in an inner product space, then S is linearly independent.

Theorem 6.3.3

If W is a finite dimensional subspace of an inner product space V, then every vector $\boldsymbol{u}$ in V can be expressed in exactly one way as

$$\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

where $\boldsymbol{w}_1$ is in W and $\boldsymbol{w}_2$ is in $W^{\perp}$.

Theorem 6.3.4

Let W be a finite dimensional subspace of an inner product space V.

- If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ is an orthonormal basis for W, and $\boldsymbol{u}$ is any vector in V, then

$$\text{proj}_W \boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 + \cdots + \langle \boldsymbol{u}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k$$

- If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ is an orthogonal basis for W, and $\boldsymbol{u}$ is any vector in V, then

$$\text{proj}_W \boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2} \boldsymbol{v}_2 + \cdots + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_k \rangle}{\|\boldsymbol{v}_k\|^2} \boldsymbol{v}_k$$

Example 6.3.4 *Compute* $\text{proj}_W \boldsymbol{u}$ *where*

1. $\boldsymbol{u} = (1, 1)$ and $W = Span\{(1, 0), (0, 1)\}$

2. $\boldsymbol{u} = (1, 1, 2)$ and $W = Span\{(1, 0, 0), (0, 1, 2)\}$

3. $\boldsymbol{u} = (1, 2, 3)$ and $W = Span\{(1, 0, 1), (0, 3, 0)\}$

4. $\boldsymbol{u} = (2, -4, 1)$ and $W = Span\{(1, -1, 0), (1, 1, 2)\}$

Theorem 6.3.5: The Gram-Schmidt Process

Every nonzero finite-dimensional inner product space has orthonormal basis.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$ be a basis for a finite-dimensional inner product space V. The following sequence of steps will produce an orthogonal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ for V.

Step 1. Let $\boldsymbol{u}_1 = \boldsymbol{v}_1$.

Step 2. Setting $W_1 = Span\{\boldsymbol{v}_1\}$. Then

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \text{proj}_{W_1}\boldsymbol{u}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

Step 3. Setting $W_2 = Span\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Then

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \text{proj}_{W_2}\boldsymbol{u}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

Step 4. Setting $W_3 = Span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Then

$$\boldsymbol{v}_4 = \boldsymbol{u}_4 - \text{proj}_{W_3}\boldsymbol{u}_4 = \boldsymbol{u}_4 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3 \rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$$

Continuting in thus way, we will obtain, after n stepa, an orthogonal set vectors, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$.

Example 6.3.5 *Applying the Gram-Schmidt process to transform the basis B into an orthogonal basis for V.*

1. $B = \{(1, 2), (1, 1)\}$ for $\mathbb{R}^2$

2. $B = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ for $\mathbb{R}^3$

Exercise 6.3

1. Which of the following sets of vectors are orthogonal or/and orthonormal with respect to the Euclidean inner product.

   1.1 $(0,1)$ and $(2,1)$

   1.2 $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

   1.3 $(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}})$

   1.4 $(\frac{1}{3},\frac{2}{3},\frac{2}{3})$, $(\frac{2}{3},-\frac{1}{3},\frac{2}{3})$ and $(-\frac{1}{3},\frac{2}{3},\frac{2}{3})$

2. Let $\mathbb{R}^2$ have the Euclidean inner product, and let $B=\{(\frac{3}{5},-\frac{4}{5}),(\frac{4}{5},\frac{3}{5})\}$. Find the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that have coordinate vectors $[\boldsymbol{u}]_B=(1,1)$ and $[\boldsymbol{v}]_B=(-1,5)$

3. Let $\mathbb{R}^3$ have the Euclidean inner product, and let $B=\{(2,1,0),(0,0,1),(-1,2,0)\}$. Find the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that have coordinate vectors $[\boldsymbol{u}]_B=(1,1,2)$ and $[\boldsymbol{v}]_B=(-1,2,5)$

4. Compute $\text{proj}_W\boldsymbol{u}$ where

   4.1 $\boldsymbol{u}=(1,-1)$ and $W=Span\{(1,0),(0,-2)\}$

   4.2 $\boldsymbol{u}=(2,3,2)$ and $W=Span\{(1,1,0),(0,1,2)\}$

   4.3 $\boldsymbol{u}=(1,2,-1)$ and $W=Span\{(1,1,1),(0,2,0)\}$

   4.4 $\boldsymbol{u}=(3,4,1)$ and $W=Span\{(1,-3,0),(1,1,1)\}$

5. Applying the Gram-Schmidt process to transform the basis B into an orthogonal basis

   5.1 $B=\{(3,4),(4,3)\}$ for $\mathbb{R}^2$

   5.2 $B=\{(1,2,1),(0,1,1),(1,0,1)\}$ for $\mathbb{R}^3$

   5.3 $B=\{(1,0,1),(1,1,0),(0,0,2)\}$ for $\mathbb{R}^3$

Chapter 7

Eigenvalues and Eigenvectors

7.1 Eigenvalues and Eigenvectors

Definition 7.1.1: Eigenvalues and Eigenvector

Let A be an $n \times n$ matrix . Then a nonzero vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\boldsymbol{x}$ is said to be an eigenvector of A corresponding to λ.

Example 7.1.1 *Determine whether $\boldsymbol{x}$ is an eigenvector of A.*

1. $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

2. $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix}$

3. $\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

4. $\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

To find the eigenvalue of an $n \times n$ matrx A we rewrite $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as

$$\begin{aligned} A\boldsymbol{x} &= \lambda I\boldsymbol{x} \\ A\boldsymbol{x} - \lambda I\boldsymbol{x} &= \mathbf{0} \\ (A - I\lambda)\boldsymbol{x} &= \mathbf{0} \end{aligned}$$

Thus, This equation has a nonzero solution if and only if

$$\det(A - \lambda I) = 0$$

this is called the **characteristic equation** of A; the scalar satisfying this equation are eigenvalues of A. The determinant $\det(A - \lambda I)$ is a polynomial in λ called the **characteristic polynomial** of A.

$$\det(A - \lambda I) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

Example 7.1.2 *Find the eigenvalues of A*

1. $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 \\ 9 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Example 7.1.3 *Find the eigenvalues of A*

1. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Theorem 7.1.1

If A is an $n \times n$ triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A.

Theorem 7.1.2: Equivalent Statements

If A is an $n \times n$ matrix λ is a real number, then TFAE.

1. λ is an eigenvalue of A.
2. The system of equations $(A - \lambda I)\boldsymbol{x} = 0$ has nontrivial solutions.
3. There is a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$.
4. λ is a solution of the characteristic equation $\det(A - \lambda I) = 0$.

The eigenvectors of A corresponding to an eigenvalue λ are the nonzero vectors $\boldsymbol{x}$ such that satisfy

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

The solution space of $\boldsymbol{x}$ is called **eigenspace** of A corresponding to λ.

Example 7.1.4 *Find the basis of the eigenspaces of* $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Example 7.1.5 *Find the eigenspace of A*

1. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

Theorem 7.1.3

If k is a positive integer, λ is an eigenvalue of A, and $\boldsymbol{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\boldsymbol{x}$ is a corresponding eigenvector.

Example 7.1.6 *Compute eigenvalues of* A^5 *where* $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

Theorem 7.1.4

A square matrix is invertible if and only if all eigenvalue are not zero.

Definition 7.1.2

Let T be a linear operator on a vector space V. Then a nonzero vector $\boldsymbol{x} \in V$ is called an **eigenvector** of T if

$$T(\boldsymbol{x}) = \lambda \boldsymbol{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and $\boldsymbol{x}$ is said to be an eigenvector of T corresponding to λ.

For $T_A : \mathbb{R}^n \to \mathbb{R}^n$, the eigenvalues of T_A can compute by $A\boldsymbol{x} = \lambda \boldsymbol{x}$ or $\det(A - \lambda I) = 0$.

Example 7.1.7 *Find eigenvalues and eigenspaces of T*

1. $T(x, y) = (x + 4y, x + y)$

2. $T(x, y) = (2y, -x + 3y)$

3. $T(x, y, z) = (x, x + 2y, -3x + 5y + 2z)$

4. $T(x, y, z) = (3x - y, -x + 2y - z, -y + 3z)$

Exercise 7.1

1. Find the eigenvalues and eigenspaces of the following matrices:

1.1 $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ 1.2 $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$ 1.3 $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$ 1.4 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2. Find the eigenvalues and eigenspaces of the following matrices:

2.1 $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ 2.3 $\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$ 2.5 $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

2.2 $\begin{bmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ 2.4 $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$ 2.6 $\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

3. Find the characteristic equations of the following matrices:

3.1 $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 3.2 $\begin{bmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

4. Find the eigenvalue of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

5. Find eigenvalues and eigenspaces of T

5.1 $T(x, y) = (2x + 4y, x + 2y)$

5.2 $T(x, y) = (-3y, 2x + 5y)$

5.3 $T(x, y, z) = (-x + 7y - z, y, 15y + 2z)$

5.4 $T(x, y, z) = (-x + z, -x + 3y, -4x + 13y - z)$

7.2 Diagonalization

Definition 7.2.1: diagonalizable

A square matrix A is called **diagonalizable** if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to **diagonalize** A

Theorem 7.2.1

If A is an $n \times n$ matrix, then TFAE.

1. A is diagonalizable.

2. A has n linearly independent eigenvectors.

Procedure for Diagonalizing a Matrix

Step 1. Find n linearly independent eigenvectors of A, say $\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_n$.

Step 2. Form the matrix P having $\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_n$ as tis column vectors.

Step 3. The matrix $P^{-1}AP$ will then diagonal with $\lambda_1, \lambda_2, ..., \lambda_n$ as its successive diagonal entries, where λ_i is the eogenvaule corresponding to $\boldsymbol{p}_i$, for $i = 1, 2, ..., n$.

Example 7.2.1 *Find a matrix P that diagonalizes*

1. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

Example 7.2.2 *Find a matrix P that diagonalizes*

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Example 7.2.3 *Find a matrix P that diagonalizes*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Theorem 7.2.2

If $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k$ are eigenvectors of A corresponding to distinct eigenvaluves $\lambda_1, \lambda_2, ..., \lambda_k$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_k\}$ is a linearly independent set.

Theorem 7.2.3

If an $n \times n$ matrix has n distinct eigenvaluves, then A is diagonailzable.

Example 7.2.4 *A matrix* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ *is diagonailzable or NOT.*

Let A be digonalizable. Then

$$\begin{aligned} P^{-1}AP &= D \\ A &= PDP^{-1} \\ A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\ A^3 &= (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1} \\ \therefore \quad A^k &= PD^kP^{-1} \end{aligned}$$

Example 7.2.5 *Compute* A^{10} *if* $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$

Example 7.2.6 *Compute A^{25} if $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$*

Exercise 7.2

1. Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

   1.1 Find the eigenvalues of A

   1.2 Is A diagonalizable ? Justify your conclusion.

2. Determine whether the matrix is diagonalzable.

   2.1 $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ 2.2 $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ 2.3 $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ 2.4 $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

3. Find a matrix P that diagonalizes

   3.1 $\begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$ 3.2 $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ 3.3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ 3.4 $\begin{bmatrix} -2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. Find A^{15} where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

5. Find A^{20} where $A = \begin{bmatrix} 1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Compute

   6.1 A^{100} 6.2 A^{-200} 6.3 A^{2017} 6.4 A^k