Chapter 4-1
ALGEBRA , EQUATIONS AND INEQUALITIES

Outlines

01

Introduction to Algebra

Intro to Algebra

Algebra is the part of math that involves variables.

Variable → A letter that represents either a specific number or all numbers

The point is to find or use **patterns true for all numbers**.

In algebraic equations, the variable represents one or two initially unknown values, and the point is to **solve** for those specific values.

Constant → A number or a symbol that doesn't change in value

Term → A product of constants and variables, including powers of variables

$$5 \qquad x \qquad 6y^2 \qquad x^2y^6z^7$$

Coefficient → The constant factor of a term.

When no constant is written, the coefficient is 1

Expression → A collection of one or more terms joined by addition or subtraction

$x+y$	y^2-x^2	$1+x^3+x^6$
	$a^2+2ab+b^2$	
	ax^2+bx+c	

NOTICE : expressions don't have equal signs

Monomial → An expression with exactly one term

$5x^2$ $\quad 43 \quad$ $x^6y^5z^{12}$ $\quad y$

Binomial → An expression with exactly two terms

$x+5 \quad s^2+t^2 \quad x^2-6x \quad a^3+b^6$

$x+5 \quad 1037p^{75}r^{85}q^{95}-3$

Trinomial → An expression with exactly three terms

$x^2+4x-21$

$a^2+b^2-c^2$

Polynomial → An expression with any number of terms, involving only one variable

Linear → A term with a single power of a variable. (i.e. no explicitly written exponent)

Quadratic → A term with the square of a single variable

Cubic → A term with the cube of a single variable

The words linear and quadratic can describe individual terms, but they can also describe entire expressions involving a single variable.

In a linear expression, the highest power of the variable is 1.

$\mathbf{17x}$ is a linear monomial

$\mathbf{3x-5}$ is a linear binomial

A linear binomial must have one linear term and one constant term.

A linear binomial must have one linear term and one constant term. In a quadratic expression, the highest power of the variable is 2.

$14x^2$ is a quadratic monomial

$x^2 - 4$ is a quadratic binomial with a quadratic term and a constant term

$2x^2 + 8x + 1$ is a quadratic trinomial with a quadratic term, a linear term and a constant term

02

Simplifying Expressions

Simplify $3x + 5x = 8x$

We can add 3x and 5x, because they are **like terms**.

Like terms → Any two terms with the same variable part. If they differ at all, they differ only in coefficients.

$$5xy + 7yx = 12xy$$

$$6a^8b^2c^6 - 4b^2c^6a^8 = 2a^8b^2c^6$$

Practice questions. Simplify the following

$$\left(a^2-2ab+b^2\right)-\left(a^2+2ab+b^2\right)=\ldots$$

$$(2x-w)-(2w-y)+(2y-x)=\ldots$$

$$(2x-w)+(2w-y)-(2y-x)=\ldots$$

03

Multiplying Expressions

Multiplying Expressions

In this lesson, we will begin the discussion of multiplying expressions, looking at two relatively simple cases: monomial times monomial, and monomial times binomial.

Remember that multiplication is **commutative & associative.** This means that we can put factors in any orders and/or groupings and not change the product.

$$ABCD = BADC = (AD)(BC) = (C)(BDA) = etc.$$

Multiplying Expressions

Also, we need to be clear on the Distributive Law.

Multiplication distributes over addition and subtraction:

$$A(B+C)=AB+AC$$

$$A(B-C)=AB-AC$$

Multiplication does **NOT** distribute over multiplication!!

$$3\times(xy)\neq(3x)(3y)\rightarrow 3\times(xy)=3xy$$

Multiplying Expressions

3.1 Multiplying two monomials

If we multiply a number, a constant, times a monomial with a variable, the constant multiplies the coefficient.

$$7 \times \left(5x^2\right) = 35x^2$$

$$2 \times \left(r^4 s^2 t^3\right) = 2r^4 s^2 t^3$$

Remember that dividing by a number is the same as multiplying by its reciprocal.

$$\frac{15x^6 y^{12}}{3} = 15x^6 y^{12} \times \frac{1}{3} = 5x^6 y^{12}$$

Multiplying Expressions

For higher powers of x, recall the rule for multiplying powers:

$$\boxed{\left(x^a\right)\left(x^b\right)=x^{a+b}}$$

<u>For example</u>

$$\left(3x\right)\left(4x^2\right)=12x^3$$

$$\left(7x^2y^2\right)\left(6xy^3\right)=42x^3y^5$$

The powers of different variables stay separate.

Multiplying Expressions

Practice problems:

$$\left(5x^5\right)\left(6x^6\right) \quad = \quad \ldots$$

$$\left(3x^3y^2\right)\left(7x^5y^6\right) \quad = \quad \ldots$$

$$\left(2xy\right)\left(3xz\right)\left(4yz\right) \quad = \quad \ldots$$

Multiplying Expressions

3.2 A monomial times a binomial

For this, we use the Distributive Law :

$$\boxed{A(B+C)=AB+AC}$$

$$7x^2(x^3+2x^2)=(7x^2)(x^3)+(7x^2)(2x^2)$$
$$=7x^5+14x^4$$

Cleary, we could extend this pattern for a monomial times a trinomial or any higher polynomial.

04

The FOIL Method

The FOIL Method

Consider

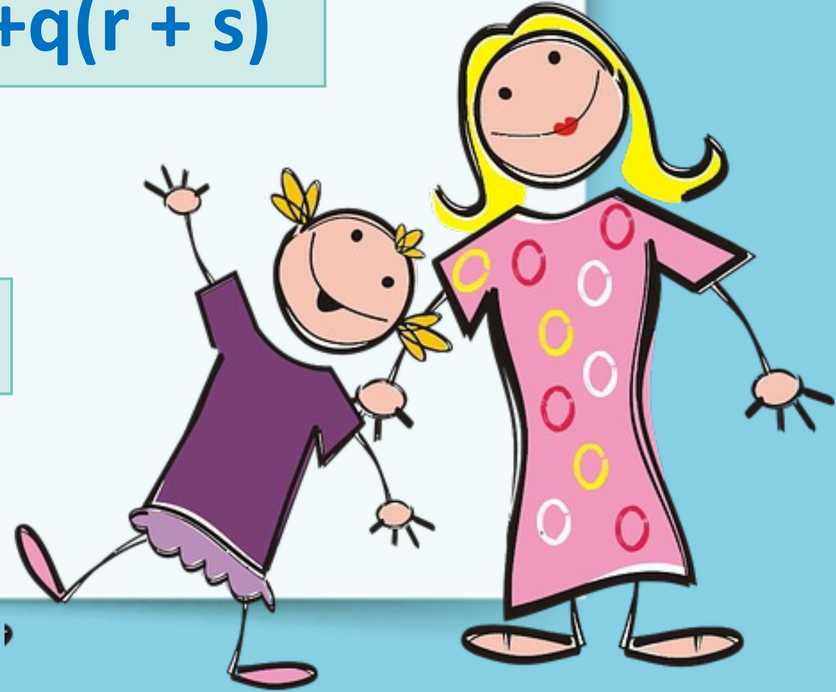
$$(p + g)(r + s) = ?$$

Technically, we would have to distribute one factor:

$$(p + q)(r + s) = p(r + s) + q(r + s)$$

then distribute in each term:

$$pr + ps + qr + qs$$



The FOIL Method

You can think of this process using the Distributive Law in that way, but there's a very convenient shortcut, summarized by the mnemonic **FOIL**.

F = First	$(p + q)(r + s)$
O = Outer	$(p + q)(r + s)$
I = Inner	$(p + q)(r + s)$
L = Last	$(p + q)(r + s)$

The product of the binomials is the sum of those four individual products.

The FOIL Method

Here's an example of the use of **FOIL**.

$$(2x + y)(x + 2y)$$

First:	$(2x)(x)$	$= 2x^2$
Outer:	$(2x)(2y)$	$= 4xy$
Inner:	$(Y)(x)$	$= xy$
Last:	$(y)(2y)$	$= 2y^2$

$$(2x + y)(x + 2y) = 2x^2 + 4xy + xy + 2y^2$$

$$= 2x^2 + 5xy + 2y^2$$

The FOIL Method

Another example of FOIL

$$(2x - 5)(3x - 1) = 6x^2 - 2x - 15x + 5$$

$$= 6x^2 - 17x + 5$$

$$(x^4 + x)(x^5 + x) = x^9 + x^6 + x^6 + x^3$$

$$= x^9 + 2x^6 + x^3$$

The FOIL Method

Practice problems:

$(x - 5)(x + 7)$ =........................

$(x - 4)(x - 6)$ =........................

$(x^2 + 2)(x + 2)$ =........................

The FOIL Method

A very common algebraic mistake pattern.

$$(a + b)^2 \neq a^2 + b^2$$

It is absolutely illegal to "distribute" an exponent across addition or subtraction.

$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

This rule: $(a + b)^2 = a^2 + 2ab + b^2$ is called the **Square of a Sum**.

The FOIL Method

A similar rule, the **Square of a Difference**, can also be found by using FOIL:

$$(a - b)^2 \quad = a^2 - 2ab + b^2$$

These two formulas, the **Square of a Sum** and the **Square of a Difference**, are good to remember, because they are two of the most important patterns in all of algebra.

05

Factoring - GCF

Factoring - GCF

In the previous two lessons, we made extensive use of the Distributive Law.

$$\boxed{P(Q + R) = PQ + PR}$$

When we start with the left side and move to the right, that is called **"distributing".**

Factoring - GCF

Factoring out a GCF from a binomial

$5x + 45$	$= 5(x + 9)$
$9x^3 + 12x$	$= 3x(3x^2 + 4)$
$x^7y^4 + x^5y^6$	$= x^5y^4(x^2 + y^2)$

We could also factor out a GCF from a trinomial

$$2x^4 + 2x^3 + 10x^2 = 2x^2(x^2 + x + 5)$$

Factoring - GCF

Practice problems

$13x^2 + 26x$ =.........................

$15xy - 18xy^2$ =.........................

$x^5 - 8x^4 + 15x^3$ =.........................

06

Factoring-Difference of Two Squares

Factoring-Difference of Two Squares

This lesson discusses perhaps the single most important factoring pattern in algebra.

We already learned about the Square of a Sum and the Square of a Difference. Those two, with this new one, are the "big three" factoring patterns, and this last one is clearly the most important.

Factoring-Difference of Two Squares

This pattern is known as the **Difference of Two Squares**.

$$a^2 - b^2 = (a + b)(a - b)$$

It's easy to see why this is true by using FOIL on the right side.

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

Factoring-Difference of Two Squares

This pattern allows us to factor many different expressions.

$x^2 - 49$	$= (x + 7)(x-7)$
$9x^2 - 16$	$= (3x - 4)(3x + 4)$
$25x^2 - 64y^2$	$= (5x + 8y)(5x - 8y)$
$x^2y^2 - 1$	$= (xy - 1)(xy + 1)$

Factoring-Difference of Two Squares

The material covered by this pattern expands considerably when we realize this: any even power of x is the square of another power.

First, we can express any even integer as the sum of two of the same integer.

$$10 = 5 + 5$$
$$18 = 9 + 9$$

This means we could split any even exponent into two powers with equal exponents.

$$x^{10} = x^{5+5} = x^5x^5 = (x^5)^2$$

Factoring-Difference of Two Squares

We can also see this from the power-to-a-power rule:

$$x^{10} = x^{5*2} = (x^5)^2$$

Either way, the even power of a variable is the square of the power with half the exponent.

$$x^6 = (x^3)^2$$
$$x^8 = (x^4)^2$$
$$x^{12} = (x^5)^2$$

Factoring-Difference of Two Squares

Since all even powers are squares, we can use the Difference of Two Squares pattern to factor expressions involving even powers.

$x^6 - 16$	$= (x^3 - 4)(x^3 + 4)$
$x^8 - 9y^2$	$= (x^4 + 3y)(x^4 - 3y)$
$x^7 - 4x^5$	$= x^5(x^2 - 4)$
	$= x^5(x - 2)(x + 2)$
$x^4 - 81$	$= (x^2 + 9)(x^2 - 9)$
	$= (x^2 + 9)(x - 3)(x + 3)$

Factoring-Difference of Two Squares

It is worth noting that there is no way to factor the sum of two squares.

The big three patterns to know are:

1) The Square of a Sum: $(a + b)^2 = a^2 + 2ab + b^2$
2) The Square of a Difference: $(a - b)^2 = a^2 - 2ab + b^2$
3) The Difference of Two Squares: $(a + b)(a - b) = a^2 - b^2$

Factoring-Difference of Two Squares

Practice problems

$y^2 - 25$ = ...

$25s^2 - 14$ = ...

$x^9 - x$ = ...

If $y = 5 + x$ and $y = 12 - X$, and if $y^2 = x^2 + K$, then K equals which of the following?

A) 17 B) 25

C) 60 D)119

07

Factoring-Quadratics

Factoring-Quadratics

The most challenging topic in factoring that you need to know for the test involves factoring a quadratic trinomial into the product of two linear binomials.

$$x^2 - bx + c = (x + p)(x - q)$$

For example, on the test we might have to factor something like

$$x^2 + 4x - 21$$

Factoring-Quadratics

Let's think about the details of the FOIL process. Suppose we start with two generic linear binomials:

$$(x + p)(x + q) \quad = x^2 + qx + px + pq$$
$$= x^2 + (p + q)x + pq$$

We need to find the unknown constants p & q. The linear coefficient of the quadratic is **SUM** of these two unknowns, and the constant term of the quadratic is the **PRODUCT** of the two unknowns.

Factoring-Quadratics

Factor: $x^2 + 8x + 15$

We need to find two numbers whose sum = 8 and whose product = 15. Clearly, those numbers are 3 & 5.

$$x^2 + 8x + 15 = (x + 3)(x + 5)$$

Factor: $X^2 + 14x + 24$

sum = 14, product = 24 two numbers: 2 and 12

$$x^2 + 14x + 24 = (x + 2)(x + 12)$$

Factoring-Quadratics

Factor: $x^2 + 4x - 21$

sum = 4, product = -21

Since the product is negative, one must be positive and one negative and the sum is positive, the larger number must be positive numbers = +7 and -3

$$x^2 + 4x - 21 = (x + 7)(x - 3)$$

Factor: $x^2 - 2x - 35$

sum = -2, product = -35 one negative, one positive; one with bigger absolute value is negative numbers = -7 and +5

$$x^2 - 2x - 35 = (x - 7)(x + 5)$$

Factoring-Quadratics

Factor: $x^2 - 16x + 48$

sum = -16, product = +48 both numbers negative numbers = -12 and -4

$$x^2 - 16x + 48 = (x - 12)(x - 4)$$

Factor:

$x^2 + 6x + 5 = (x + 5)(x + 1)$

$x^2 - 5x - 14 = (x - 7)(x + 2)$

$x^2 + 6x - 27 = (x + 9)(x - 3)$

$x^2 - 20x + 36 = (x - 2)(x - 18)$

Factoring-Quadratics

<u>Notice</u> that this method only works if the coefficient of x is 1.

If the quadratic coefficient is something other than 1, the chances are very good that one of the other factoring methods can be used (Difference of Two Squares, GCF, etc.)

08

Factoring- Combined

Factoring- Combined

Now that we have learned a few different factoring techniques, we can explore methods for combining them to factor more challenging expressions.

These combined techniques would be applicable only on the most difficult Quant problems on the test.

Factoring- Combined

Factor: $6x^3 - 150x = 6x(x^2 - 25)$
$= 6x(x + 5)(x - 5)$

Factor: $2x^2 - 22x + 48 = 2(x^2 - 11x + 24)$
$= 2(x - 8)(x - 3)$

Factor: $7x^4 - 56x^3 - 63x^2 = 7x^2(x^2 - 8x - 9)$
$= 7x^2(x - 9)(x + 1)$

Factor: $-3x^9 + 48xy^4 = 48xy^4 - 3x^9$
$= 3x(16y^4 - x^8)$
$= 3x(4y^2 + x^4)(4y^2 - x^4)$

09

Advanced Numerical Factoring

Advanced Numerical Factoring

In this lesson, we will return to the idea of prime factorization from the **INTEGER PROPERTIES** module.

Now that we have learned these algebraic factoring techniques, including the Difference of Two Squares, we can use some of these to factor numbers that otherwise would be very hard to factor.

Advanced Numerical Factoring

Example 1

Find the prime factorization of 1599.

Notice that :

$$\begin{aligned}1599 = 1600 - 1 &= \left(40^2\right) - \left(1^2\right)\\ &= (40+1)(40-1)\\ &= (41)(39)\\ &= (41)(3*13)\\ &= 41*3*13\end{aligned}$$

Example 2

Find the prime factorization of 2491.

Notice that :

$$\begin{aligned}2491 = 2500 - 9 &= (50^2) - (3^2)\\ &= (50 - 3)\,(50 + 3)\\ &= 47*53\end{aligned}$$

Advanced Numerical Factoring

Find the prime factorization of 9975.

Notice that :

$$9975 = 10000 - 25 = (100^2) - (5^2)$$
$$= (100 + 5)\,(100 - 5)$$
$$= (105)\,(95)$$
$$= (5*21)\,(5*19)$$
$$= (5*3*7)\,(5*19)$$
$$= 5^2*3*7*19$$

Advanced Numerical Factoring

Practice Problem

1. $\frac{0.999951}{0.993} = \ldots$

2. $\left(\frac{0.999856}{0.988} - 1\right) = \ldots$

10

Factoring - Rational Expressions

Factoring - Rational Expressions

In this lesson, we will examine rational expressions.

A rational expression is **a ratio**, **a fraction**, of two algebraic expressions.

$$\frac{x - 3}{x + 2}$$

Factoring - Rational Expressions

Many of the rational expressions you will see on the test will be ones that can be simplified, and your job will be to simplify them.

To simplify a rational expression, we have to factor the expressions in the numerator and in the denominator, and then cancel common factors.

Factoring - Rational Expressions

Simplify:

$$\frac{x^2 + 4x - 21}{x^2 - 6x + 9} = \frac{(x-3)(x+7)}{(x-3)^2} = \frac{(x+7)}{(x-3)}$$

Factoring - Rational Expressions

Example

Simplify:

$$\frac{2x^4-8x^2}{x^2-5x-14}=\frac{2x^2\left(x^2-4\right)}{x^2-5x-14}$$

$$=\frac{2x^2(x+2)(x-2)}{(x+2)(x-7)}$$

$$=\frac{2x^2(x-2)}{(x-7)}$$

Simplify:

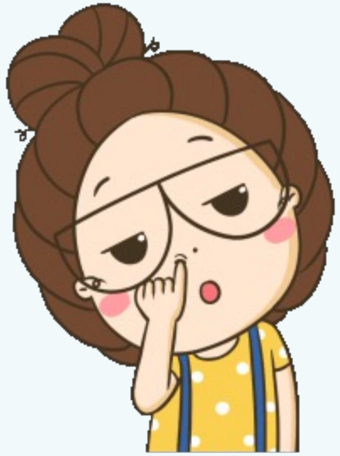
$$\frac{y^2+2x-8}{x-4}=\frac{y^2}{x-4}+\frac{2x-8}{x-4}$$

$$=\frac{y^2}{x-4}+\frac{2(x-4)}{x-4}$$

$$=\frac{y^2}{x-4}+2$$

Factoring - Rational Expressions

Practice question



$$\frac{x^2+12x-45}{x+15}=22\text{, then } x=?$$

$$\frac{4P+3Q-4R}{P-R}=19\text{, then } \frac{Q}{P-R}=?$$