Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Midterm Examination
Semester 2/2024

Course ID	**Course Name**	**Test Time**	**Full Scores**
MAC3309	Mathematical Analysis	5pm - 8pm Mon 27 Jan 2025	100 marks 25%

Name.. **ID**..................................... **Section**...............

Direction

1. 10 questions of all 12 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
4. Cannot answer sheets out of test room.
5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Scores											

Some Definition to prove this examination.

1. $\lim_{n\to\infty} x_n = a \iff \forall\varepsilon > 0\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow |x_n - a| < \varepsilon$

2. $\lim_{n\to\infty} x_n = +\infty \iff \forall M \in \mathbb{R}\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow x_n > M$

3. $\lim_{n\to\infty} x_n = -\infty \iff \forall M \in \mathbb{R}\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow x_n < M$

4. $\lim_{x\to a} f(x) = L \iff \forall\varepsilon > 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon$

5. $\lim_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0\ \exists M \in \mathbb{R},\ x > M \longrightarrow |f(x) - L| < \varepsilon$

6. $\lim_{x\to-\infty} f(x) = L \iff \forall\varepsilon > 0\ \exists M \in \mathbb{R},\ x < M \longrightarrow |f(x) - L| < \varepsilon$

7. $\lim_{x\to a} f(x) = +\infty \iff \forall M > 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow f(x) > M$

8. $\lim_{x\to a} f(x) = -\infty \iff \forall M < 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow f(x) < M$

9. E is open $\iff \forall x \in E\ \exists\delta > 0,\ (x - \delta, x + \delta) \subseteq E$

10. F is closed $\iff F^c = \mathbb{R} - F$ is open

11. x is a limit point of A $\iff \forall\varepsilon > 0\ [(x - \varepsilon, x) \cup (x, x + \varepsilon)] \cap A \neq \varnothing$

1. **(10 marks)** Let a, b and c be real numbers. Prove that

$$\left(\frac{a+b+c}{3}\right)^2 \leq \left(\frac{a^2+b^2+c^2}{3}\right).$$

Hint : Use the perfect square of three numbers, $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$.

2. **(10 marks)** Let x and y be real numbers. Prove that

$$\text{if} \quad x + |y| = |x| + y, \quad \text{then} \quad xy \geq 0.$$

3. **(10 marks)** Define the set

$$A = \left\{ \frac{21}{2n+5} : n \in \mathbb{N} \right\}.$$

Find **sup** $\boldsymbol{A}$ and **inf** $\boldsymbol{A}$ with proving them.

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{6n^2+8}{2n^2+5} = 3.$$

5. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{n^2+5}{n+5} = +\infty.$$

6. **(10 marks)** Use definition to prove that

$$\left\{\frac{n}{n^2+1}\right\} \text{ is a Caucy sequence.}$$

7. **(10 marks)** Define a set

$$A = \left\{ \frac{1}{n} : n \in \ N \right\}$$

Show that 0 is a limit point of A.
Hint : Use Archimedean property.

8. **(10 marks)** Use definition to prove that

$$\lim_{x \to 2} \frac{5x}{x^2 + 1} = 2.$$

9. **(10 marks)** Use definition prove that

$$\lim_{x \to 1^-} \frac{x}{1-x} = +\infty.$$

10. **(10 marks)** Let $\{x_n\}$ and $\{y_n\}$ be sequences in real.
Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$. Prove that

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Solution Midterm Exam. 2/2023
MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

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1. **(10 marks)** Let a, b and c be real numbers. Prove that

$$\left(\frac{a+b+c}{3}\right)^2 \leq \left(\frac{a^2+b^2+c^2}{3}\right).$$

Hint : Use the perfect square of three numbers, $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$.

Proof. Let a, b and c be real numbers. By the fact that

$$(a-b)^2 \geq 0,\ (b-c)^2 \geq 0 \text{ and } (c-a)^2 \geq 0.$$

We obtain

$$\begin{aligned}
0 &\leq (a-b)^2 + (b-c)^2 + (c-a)^2 \\
0 &\leq (a^2-2ab+b^2) + (b^2-2bc+c^2) + (c^2-2ac+a^2) \\
0 &\leq 2a^2+2b^2+2c^2-2ab-2bc-2ac \\
2ab+2bc+2ac &\leq 2a^2+2b^2+2c^2 \\
2ab+2bc+2ac+(a^2+b^2+c^2) &\leq 2a^2+2b^2+2c^2+(a^2+b^2+c^2) \\
a^2+b^2+c^2+2ab+2bc+2ac &\leq 3a^2+3b^2+3c^2 \\
(a+b+c)^2 &\leq 3(a^2+b^2+c^2) \\
\frac{(a+b+c)^2}{9} &\leq \frac{3(a^2+b^2+c^2)}{9} \\
\left(\frac{a+b+c}{3}\right)^2 &\leq \left(\frac{a^2+b^2+c^2}{3}\right)
\end{aligned}$$

□

2. **(10 marks)** Let x and y be real numbers. Prove that

$$\text{if} \quad x + |y| = |x| + y, \quad \text{then} \quad xy \geq 0.$$

Proof. Let x and y be real numbers.
Assume that $x + |y| = |x| + y$. Then

$$\begin{aligned} |y| - |x| &= y - x \\ (|y| - |x|)^2 &= (y - x)^2 \\ |x|^2 - 2|x||y| + |y|^2 &= x^2 - 2xy + y^2 \\ x^2 - 2|xy| + y^2 &= x^2 - 2xy + y^2 \\ |xy| &= xy. \end{aligned}$$

By the definition of absolute value, we conclude that

$$xy \geq 0.$$

□

3. **(10 marks)** Define the set

$$A = \left\{ \frac{21}{2n+5} : n \in \mathbb{N} \right\}.$$

Find **sup A** and **inf A** with proving them.

Consider $A = \left\{ 3, \frac{7}{3}, \frac{21}{11}, \frac{21}{13}, \dots \right\}$.

Claim that $\inf A = 0$ and $\sup A = 3$

Proof. $\inf A = 0$

Let $n \in \mathbb{N}$. Then $n > 0$. So, $2n+5 > 0$. It's clear that

$$\begin{aligned} 0 &< \frac{1}{2n+5} \\ 0 \cdot 21 &< \frac{1}{2n+5} \cdot 21 \\ 0 &< \frac{21}{2n+5}. \end{aligned}$$

Thus, 0 is a lower bound of A.

Suppose that there is a lower bound ℓ_0 of A such that $\ell_0 > 6$. It follows that

$$\ell_0 \leq \frac{21}{2n+5} \quad \text{for all } n \in \mathbb{N}. \qquad (*)$$

From $\frac{2\ell_0}{21} > 0$, by Archimendean property, there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2\ell_0}{21}.$$

Since $2n_0 + 5 \geq 2n_0$,

$$\frac{21}{2n_0+5} \leq \frac{21}{2n_0} = \frac{21}{2} \cdot \frac{1}{n_0} < \frac{21}{2} \cdot \frac{2\ell_0}{21} = \ell_0.$$

So, $\frac{21}{2n_0+5} < \ell_0$. This is contradiction to $(*)$. Therefore, $\inf A = 0$.

$\sup A = 3$

Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $2n \geq 2$. We obtain $2n+5 \geq 7$ and then

$$\begin{aligned} \frac{1}{2n+5} &\leq \frac{1}{7} \\ \frac{21}{2n+5} &\leq 21 \cdot \frac{1}{7} = 3. \end{aligned}$$

Thus, 3 is an upper bound of A.

Let u be an upper bound of A. Then

$$\frac{21}{2n+5} \leq u \quad \text{for all } n \in \mathbb{N}.$$

Choose $n = 1$, we obtain

$$\frac{21}{2(1)+5} = 3 \leq u.$$

Therefore, $\sup A = 3$. □

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{6n^2+8}{2n^2+5} = 3.$$

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{7}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\dfrac{1}{n} \leq \dfrac{1}{N}$. Since $2n^2+5 > 2n^2 > n^2$, $\dfrac{1}{2n^2+5} < \dfrac{1}{n^2}$.
From $n^2 \geq n$ ($\because n \geq 1$), we have $\dfrac{1}{n^2} < \dfrac{1}{n}$. It follows that

$$\begin{aligned}\left|\frac{6n^2+8}{2n^2+5} - 3\right| &= \left|\frac{(6n^2+8) - 3(2n^2+5)}{2n^2+5}\right| = \left|\frac{6n^2+8-6n^2-15}{2n^2+5}\right| \\ &= \left|\frac{-7}{2n^2+5}\right| \\ &= \frac{7}{2n^2+5} \leq \frac{7}{2n^2} \leq \frac{7}{n^2} \leq \frac{7}{n} \leq \frac{7}{N} < \varepsilon.\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} \frac{6n^2+8}{2n^2+5} = 3$. □

5. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{n^2+5}{n+5} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that

$$M + 5 < N.$$

It's equivalent to $N - 5 > M$.
Let $n \in \mathbb{N}$ such that $n \geq N$. So, $n - 5 \geq N - 5$. Since $5 \geq -25$, $n^2 + 5 \leq n^2 - 25$. We obtain

$$\frac{n^2+5}{n+5} = (n^2+5)\cdot\frac{1}{n+5} \geq (n^2-25)\cdot\frac{1}{n+5} = \frac{(n-5)(n+5)}{n+5} = n-5 \geq N-5 > M.$$

Hence, $\displaystyle\lim_{n\to\infty} \frac{n^2+5}{n+5} = +\infty$. □

6. **(10 marks)** Use definition to prove that

$$\left\{\frac{n}{n^2+1}\right\} \text{ is a Caucy sequence.}$$

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$ and $\frac{1}{m} \leq \frac{1}{N}$.
From $1 > 0$, we have

$$n^2 + 1 > n^2 \quad \text{and} \quad m^2 + 1 > m^2.$$

So,

$$\frac{1}{n^2+1} < \frac{1}{n^2} \quad \text{and} \quad \frac{1}{m^2+1} \leq \frac{1}{m^2}.$$

It follows that

$$\begin{aligned}\left|\frac{n}{n^2+1} - \frac{m}{m^2+1}\right| &= \left|\frac{n}{n^2+1}\right| + \left|\frac{m}{m^2+1}\right| \\ &= \frac{n}{n^2} + \frac{m}{m^2} \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{1}{N} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $\left\{\frac{n}{n^2+1}\right\}$ is Cauchy.

□

7. **(10 marks)** Define a set

$$A = \left\{\frac{1}{n} : n \in \ N\right\}$$

Show that 0 is a limit point of A.
Hint : Use Archimedean property.

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon.$$

From $0 < \frac{1}{n} < \varepsilon$, we have

$$\tfrac{1}{n} \in (0, \varepsilon) \quad \text{and} \quad \tfrac{1}{n} \in A.$$

Thus, $[(-\varepsilon, 0) \cup (0, \varepsilon)] \cap A \neq \varnothing$.
We conclude that 0 is a limit point of A. □

8. **(10 marks)** Use definition to prove that

$$\lim_{x\to 2}\frac{5x}{x^2+1}=2.$$

Proof. Let $\varepsilon>0$. Choose $\delta=\min\left\{1,\dfrac{2\varepsilon}{5}\right\}$. Suppose that $0<|x-2|<\delta$. Then $0<|x-2|<1$. We have

$$\begin{aligned}
&-1<x-2<1\\
&1<x<3 \quad\longrightarrow\quad 2-1<2x-1<6-1\\
&1<x^2<9\\
&2<x^2+1<10
\end{aligned}$$

It follows that

$$|2x-1|<5 \quad\text{and}\quad \frac{1}{10}<\frac{1}{x^2+1}<\frac{1}{2}$$

So,

$$\begin{aligned}
\left|\frac{5x}{x^2+1}-2\right| &= \left|\frac{5x-2(x^2+1)}{x^2+1}\right| = \left|\frac{-(2x^2-5x+2)}{x^2+1}\right|\\
&= \frac{|2x^2-5x+2|}{x^2+1} = \frac{|(2x-1)(x-2)|}{2(x^2+1)}\\
&= |x-2|\cdot|2x-1|\cdot\frac{1}{x^2+1}\\
&< \delta\cdot 5\cdot\frac{1}{2} \le \frac{2\varepsilon}{5}\cdot\frac{5}{2}=\varepsilon.
\end{aligned}$$

Therefore, $\lim_{x\to 2}\dfrac{5x}{x^2+1}=2.$ □

9. **(10 marks)** Use definition prove that

$$\lim_{x\to 1^-} \frac{x}{1-x} = +\infty.$$

Proof. Let $M > 0$. Choose $\delta = \dfrac{1}{M+1}$. It is equivalent to

$$M = \frac{1}{\delta} - 1.$$

Let $x \in \mathbb{R}$ such that $-\delta < x - 1 < 0$. Then $0 < 1 - x < \delta$. We obatin

$$\frac{1}{1-x} > \frac{1}{\delta}$$

Then

$$\frac{x}{1-x} = -1 + \frac{1}{1-x} > -1 + \frac{1}{\delta} = M.$$

Thus, $\lim_{x\to 1^-} \dfrac{x}{1-x} = +\infty$. □

10. **(10 marks)** Let $\{x_n\}$ and $\{y_n\}$ be sequences in real.
Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$. Prove that

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Proof. Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$.
Then, there is a $K > 0$ such that

$$|x_n| \leq K \quad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$. There are $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad |y_n| = |y_n - 0| < \frac{\varepsilon}{K}$$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$\begin{aligned} |x_n y_n - 0| &= |x_n y_n| = |x_n| \cdot |y_n| \\ &< K \cdot \frac{\varepsilon}{K} = \varepsilon. \end{aligned}$$

Thus, $x_n y_n \to 0$ as $n \to \infty$. □