

Suan Sunandha Rajabhat University Faculty of Education, Division of Mathematics Midterm Examination Semester 2/2024

Direction

- 1. 10 questions of all 12 pages.
- 2. Write obviously your name, id and section all pages.
- 3. Don't take text books and others come to the test room.
- 4. Cannot answer sheets out of test room.
- 5. Deliver to the staff if you make a mistake in the test room.

Your signature

Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

Some Definition to prove this examination.

1.
$$\lim_{n \to \infty} x_n = a$$
 $\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ n \ge N \longrightarrow |x_n - a| < \varepsilon$

2.
$$\lim_{n \to \infty} x_n = +\infty$$
 $\iff \forall M \in \mathbb{R} \ \exists N \in \mathbb{N}, \ n \ge N \longrightarrow x_n > M$

3.
$$\lim_{n \to \infty} x_n = -\infty$$
 $\iff \forall M \in \mathbb{R} \ \exists N \in \mathbb{N}, \ n \ge N \longrightarrow x_n < M$

4.
$$\lim_{x \to a} f(x) = L$$
 $\iff \forall \varepsilon > 0 \; \exists \delta > 0, \; 0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon$

5.
$$\lim_{x \to \infty} f(x) = L$$
 $\iff \forall \varepsilon > 0 \ \exists M \in \mathbb{R}, \ x > M \longrightarrow |f(x) - L| < \varepsilon$

6.
$$\lim_{x \to -\infty} f(x) = L$$
 $\iff \forall \varepsilon > 0 \ \exists M \in \mathbb{R}, \ x < M \longrightarrow |f(x) - L| < \varepsilon$

7.
$$\lim_{x \to a} f(x) = +\infty$$
 $\iff \forall M > 0 \ \exists \delta > 0, \ 0 < |x - a| < \delta \longrightarrow f(x) > M$

8.
$$\lim_{x \to a} f(x) = -\infty$$
 $\iff \forall M < 0 \; \exists \delta > 0, \; 0 < |x - a| < \delta \longrightarrow f(x) < M$

9.
$$E$$
 is open $\iff \forall x \in E \ \exists \delta > 0, \ (x - \delta, x + \delta) \subseteq E$

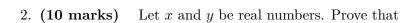
10.
$$F$$
 is closed $\iff F^c = \mathbb{R} - F$ is open

11.
$$x$$
 is a limit point of $A \iff \forall \varepsilon > 0 \ [(x - \varepsilon, x) \cup (x, x + \varepsilon)] \cap A \neq \emptyset$

1. (10 marks) Let a, b and c be real numbers. Prove that

$$\left(\frac{a+b+c}{3}\right)^2 \le \left(\frac{a^2+b^2+c^2}{3}\right).$$

Hint: Use the perfect square of three numbers, $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2xz$.



$$\text{if} \quad x+|y|=|x|+y, \quad \text{ then } \quad xy\geq 0.$$

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3. (10 marks) Define the set

$$A = \left\{ \frac{21}{2n+5} : n \in \mathbb{N} \right\}.$$

Find $\sup A$ and $\inf A$ with proving them.



$$\lim_{n \to \infty} \frac{6n^2 + 8}{2n^2 + 5} = 3.$$



$$\lim_{n \to \infty} \frac{n^2 + 5}{n + 5} = +\infty.$$

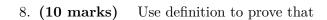
6. (10 marks) Use definition to prove that

$$\left\{\frac{n}{n^2+1}\right\}$$
 is a Caucy sequence.

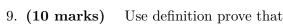
7. (10 marks) Define a set

$$A = \left\{ \frac{1}{n} : n \in N \right\}$$

Show that 0 is a limit point of A. Hint: Use Archimedean property.



$$\lim_{x \to 2} \frac{5x}{x^2 + 1} = 2.$$



$$\lim_{x \to 1^-} \frac{x}{1-x} = +\infty.$$

10. (10 marks) Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$. Prove that

$$x_n y_n \to 0 \text{ as } n \to \infty.$$



Solution Midterm Exam. 2/2023 MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

1. (10 marks) Let a, b and c be real numbers. Prove that

$$\left(\frac{a+b+c}{3}\right)^2 \le \left(\frac{a^2+b^2+c^2}{3}\right).$$

Hint: Use the perfect square of three numbers, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$.

Proof. Let a, b and c be real numbers. By the fact that

$$(a-b)^2 \ge 0$$
, $(b-c)^2 \ge 0$ and $(c-a)^2 \ge 0$.

We obtain

$$0 \le (a-b)^2 + (b-c)^2 + (c-a)^2$$

$$0 \le (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2)$$

$$0 \le 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac$$

$$2ab + 2bc + 2ac \le 2a^2 + 2b^2 + 2c^2$$

$$2ab + 2bc + 2ac + (a^2 + b^2 + c^2) \le 2a^2 + 2b^2 + 2c^2 + (a^2 + b^2 + c^2)$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \le 3a^2 + 3b^2 + 3c^2$$

$$(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$$

$$\frac{(a + b + c)^2}{9} \le \frac{3(a^2 + b^2 + c^2)}{9}$$

$$\left(\frac{a + b + c}{3}\right)^2 \le \left(\frac{a^2 + b^2 + c^2}{3}\right)$$

2. (10 marks) Let x and y be real numbers. Prove that

if
$$x + |y| = |x| + y$$
, then $xy \ge 0$.

Proof. Let x and y be real numbers. Assume that x + |y| = |x| + y. Then

$$|y| - |x| = y - x$$

$$(|y| - |x|)^2 = (y - x)^2$$

$$|x|^2 - 2|x||y| + |y|^2 = x^2 - 2xy + y^2$$

$$x^2 - 2|xy| + y^2 = x^2 - 2xy + y^2$$

$$|xy| = xy.$$

By the definition of absolute value, we conclude that

$$xy \ge 0$$
.

3. (10 marks) Define the set

$$A = \left\{ \frac{21}{2n+5} : n \in \mathbb{N} \right\}.$$

Find $\sup A$ and $\inf A$ with proving them.

Consider
$$A = \left\{3, \frac{7}{3}, \frac{21}{11}, \frac{21}{13}, \ldots\right\}$$
.

Claim that $\inf A = 0$ and $\sup A = 3$

Proof. $\inf A = 0$

Let $n \in \mathbb{N}$. Then n > 0. So, 2n + 5 > 0. It's clear that

$$0 < \frac{1}{2n+5}$$
$$0 \cdot 21 < \frac{1}{2n+5} \cdot 21$$
$$0 < \frac{21}{2n+5}.$$

Thus, 0 is a lower bound of A.

Suppose that there is a lower bound ℓ_0 of A such that $\ell_0 > 6$. It follows that

$$\ell_0 \le \frac{21}{2n+5}$$
 for all $n \in \mathbb{N}$. (*

From $\frac{2\ell_0}{21} > 0$, by Archimendean property, there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{2\ell_0}{21}.$$

Since $2n_0 + 5 \ge 2n_0$,

$$\frac{21}{2n_0+5} \le \frac{21}{2n_0} = \frac{21}{2} \cdot \frac{1}{n_0} < \frac{21}{2} \cdot \frac{2\ell_0}{21} = \ell_0.$$

So, $\frac{21}{2n_0+5} < \ell_0$. This is contradiction to (*). Therefore, $\inf A = 0$.

$\sup A = 3$

Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $2n \geq 2$. We obtain $2n + 5 \geq 7$ and then

$$\frac{1}{2n+5} \le \frac{1}{7}$$
$$\frac{21}{2n+5} \le 21 \cdot \frac{1}{7} = 3.$$

Thus, 3 is an upper bound of A.

Let u be an upper bound of A. Then

$$\frac{21}{2n+5} \le u \quad \text{ for all } n \in \mathbb{N}.$$

Choose n = 1, we obtain

$$\frac{21}{2(1)+5} = 3 \le u.$$

Therefore, $\sup A = 3$.



4. (10 marks) Use Definition to prove that

$$\lim_{n \to \infty} \frac{6n^2 + 8}{2n^2 + 5} = 3.$$

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{7}$. Let $n \in \mathbb{N}$ such that $n \ge N$. We obtain $\frac{1}{n} \le \frac{1}{N}$. Since $2n^2 + 5 > 2n^2 > n^2$, $\frac{1}{2n^2 + 5} < \frac{1}{n^2}$. From $n^2 \ge n$ (:: $n \ge 1$), we have $\frac{1}{n^2} < \frac{1}{n}$. It follows that

$$\begin{aligned} \left| \frac{6n^2 + 8}{2n^2 + 5} - 3 \right| &= \left| \frac{(6n^2 + 8) - 3(2n^2 + 5)}{2n^2 + 5} \right| = \left| \frac{6n^2 + 8 - 6n^2 - 15}{2n^2 + 5} \right| \\ &= \left| \frac{-7}{2n^2 + 5} \right| \\ &= \frac{7}{2n^2 + 5} \le \frac{7}{2n^2} \le \frac{7}{n^2} \le \frac{7}{n} \le \frac{7}{N} < \varepsilon. \end{aligned}$$

Thus, $\lim_{n \to \infty} \frac{6n^2 + 8}{2n^2 + 5} = 3$.

5. (10 marks) Use Definition to prove that

$$\lim_{n\to\infty} \frac{n^2+5}{n+5} = +\infty.$$

Proof. Let $M \in \mathbb{R}$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that

$$M + 5 < N$$
.

It's equivalent to N-5 > M.

Let $n \in \mathbb{N}$ such that $n \geq N$. So, $n-5 \geq N-5$. Since $5 \geq -25$, $n^2+5 \leq n^2-25$. We obtain

$$\frac{n^2+5}{n+5} = (n^2+5) \cdot \frac{1}{n+5} \ge (n^2-25) \cdot \frac{1}{n+5} = \frac{(n-5)(n+5)}{n+5} = n-5 \ge N-5 > M.$$

Hence, $\lim_{n\to\infty} \frac{n^2+5}{n+5} = +\infty$.

6. (10 marks) Use definition to prove that

$$\left\{\frac{n}{n^2+1}\right\}$$
 is a Caucy sequence.

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$ and $\frac{1}{m} \leq \frac{1}{N}$. From 1 > 0, we have

$$n^2 + 1 > n^2$$
 and $m^2 + 1 > m^2$.

So,

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$
 and $\frac{1}{m^2+1} \le \frac{1}{m^2}$.

It follows that

$$\begin{split} \left| \frac{n}{n^2 + 1} - \frac{m}{m^2 + 1} \right| &= \left| \frac{n}{n^2 + 1} \right| + \left| \frac{m}{m^2 + 1} \right| \\ &= \frac{n}{n^2} + \frac{m}{m^2} \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{1}{N} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{split}$$

Thus, $\left\{\frac{n}{n^2+1}\right\}$ is Cauchy.

7. (10 marks) Define a set

$$A = \left\{ \frac{1}{n} : n \in \ N \right\}$$

Show that 0 is a limit point of A.

 ${\it Hint}$: Use Archimedean property.

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon$$
.

From $0 < \frac{1}{n} < \varepsilon$, we have

$$\frac{1}{n} \in (0, \varepsilon)$$
 and $\frac{1}{n} \in A$.

Thus, $[(-\varepsilon, 0) \cup (0, \varepsilon)] \cap A \neq \emptyset$.

We conclude that 0 is a limit point of A.

8. (10 marks) Use definition to prove that

$$\lim_{x \to 2} \frac{5x}{x^2 + 1} = 2.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{2\varepsilon}{5}\right\}$. Suppose that $0 < |x-2| < \delta$. Then 0 < |x-2| < 1. We have

$$-1 < x - 2 < 1$$

$$1 < x < 3 \longrightarrow 2 - 1 < 2x - 1 < 6 - 1$$

$$1 < x^{2} < 9$$

$$2 < x^{2} + 1 < 10$$

It follows that

$$|2x-1| < 5$$
 and $\frac{1}{10} < \frac{1}{x^2+1} < \frac{1}{2}$

So,

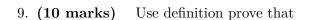
$$\left| \frac{5x}{x^2 + 1} - 2 \right| = \left| \frac{5x - 2(x^2 + 1)}{x^2 + 1} \right| = \left| \frac{-(2x^2 - 5x + 2)}{x^2 + 1} \right|$$

$$= \frac{|2x^2 - 5x + 2|}{x^2 + 1} = \frac{|(2x - 1)(x - 2)|}{2(x^2 + 1)}$$

$$= |x - 2| \cdot |2x - 1| \cdot \frac{1}{x^2 + 1}$$

$$< \delta \cdot 5 \cdot \frac{1}{2} \le \frac{2\varepsilon}{5} \cdot \frac{5}{2} = \varepsilon.$$

Therefore, $\lim_{x\to 2} \frac{5x}{x^2+1} = 2$.



$$\lim_{x \to 1^{-}} \frac{x}{1 - x} = +\infty.$$

Proof. Let M > 0. Choose $\delta = \frac{1}{M+1}$. It is equivalent to

$$M = \frac{1}{\delta} - 1.$$

Let $x \in \mathbb{R}$ such that $-\delta < x - 1 < 0$. Then $0 < 1 - x < \delta$. We obtain

$$\frac{1}{1-x} > \frac{1}{\delta}$$

Then

$$\frac{x}{1-x} = -1 + \frac{1}{1-x} > -1 + \frac{1}{\delta} = M.$$

Thus, $\lim_{x \to 1^-} \frac{x}{1-x} = +\infty$.

10. (10 marks) Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$. Prove that

$$x_n y_n \to 0 \text{ as } n \to \infty.$$

Proof. Assume that $\{x_n\}$ is bounded and $y_n \to 0$ as $n \to \infty$. Then, there is a K > 0 such that

$$|x_n| \le K$$
 for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. There are $N \in \mathbb{N}$ such that

$$n \ge N$$
 implies $|y_n| = |y_n - 0| < \frac{\varepsilon}{K}$

Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain

$$|x_n y_n - 0| = |x_n y_n| = |x_n| \cdot |y_n|$$

 $< K \cdot \frac{\varepsilon}{K} = \varepsilon.$

Thus, $x_n y_n \to 0$ as $n \to \infty$.