Math SSRU

Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Final Examination (Set A)
Semester 2/2024

Course ID	Course Name	Test Time	Full Scores
MAC3309	Mathematical Analysis	5 p.m. - 8 p.m. Fir 28 Mar. 2025	100 marks 25%

Name.. **ID**.................................... **Section**...............

Direction

1. 10 questions of all 12 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
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5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

Some Definition to prove this examination.

1. f is continuous at a $\iff$ $\forall\varepsilon > 0\ \exists\delta > 0,\ |x-a| < \delta \longrightarrow |f(x)-f(a)| < \varepsilon$

2. f is uniformly continuous on E $\iff$ $\forall\varepsilon > 0\ \exists\delta > 0\ \forall x, a \in E,\ |x-a| < \delta \longrightarrow |f(x)-f(a)| < \varepsilon$

3. f is differentiable at a $\iff$ $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists

4. f is increasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \longrightarrow f(x_1) < f(x_2)$

5. f is decreasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \longrightarrow f(x_1) > f(x_2)$

6. f is integrable on $[a,b]$ $\iff$ $\forall\varepsilon > 0\ \exists P_\varepsilon,\ U(f,P) - L(f,P) < \varepsilon$

7. Riemann sum converges to $I(f)$ $\iff$ $\forall\varepsilon > 0\ \exists P_\varepsilon \subseteq \{x_0, x_1, ..., x_n\} \longrightarrow \left|\sum_{i=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \varepsilon$

8. **Cauchy Criterion**: $\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\forall\varepsilon > 0\ \exists N \in \mathbb{N}, m > n \geq N \longrightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

1. **(10 marks)** Use definition to prove that

$$f(x) = (x^2 - 1) + (x - 1)^2$$

is continuous at $x = 1$.

2. **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = x + f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\frac{1}{x} \leq \sqrt{x} \quad \text{for all} \quad x \geq 1.$$

4. **(10 marks)** Define $f(x) = x - e^{-x}$ where $x \in \mathbb{R}$.

   Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

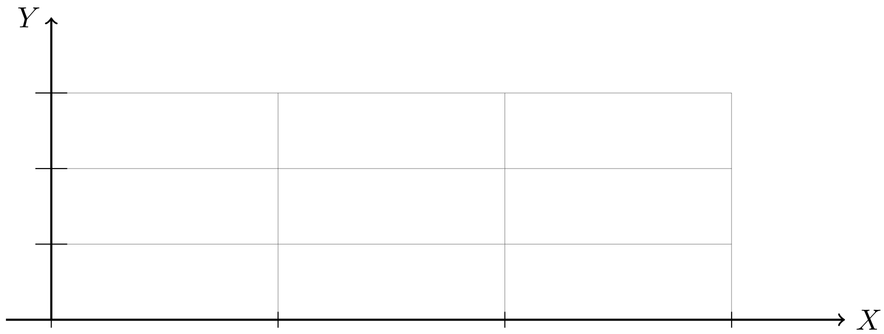
   Q2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

   Q3 **(3 marks)** Compute $(f^{-1})'(-1)$.

5. **(10 marks)** Define

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \\ 3 & \text{if } x = 1 \end{cases}$$

Draw the graph of f on $[0,2]$ and use definition to show that f is integrable on $[0,2]$.



6. **(10 marks)** Let $f(x) = (x^2 - 1) + (x-1)^2$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\} = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., 1 \right\}$$

be a partition of $[0,1]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$ on $[0,1]$.

7. **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{-x} 2g(t^2)\,dt \quad \text{where } x \in \mathbb{R}.$$

Show that $\displaystyle\int_{-1}^{0} f(x)\,dx + \int_0^1 g(x)\,dx = 0.$

Hint: Use integration by part to $\displaystyle\int_{-1}^{0} f(x)\,dx$ and change variable.

8. **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right]=\frac{5}{2}.$$

Hint: Use Telescoping and Geometric Series.

9. **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k} \right)^k \text{ converges or NOT.}$$

Verify your answer.

10. **(10 marks)** Determine whether

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^2}\right)$$

is conditionally convergent or NOT.

Math SSRU

Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Final Examination (Set B)
Semester 2/2024

Course ID	**Course Name**	**Test Time**	**Full Scores**
MAC3309	Mathematical Analysis	5 p.m. - 8 p.m. Fir 28 Mar. 2025	100 marks 25%

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

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Some Definition to prove this examination.

1. f is continuous at a $\iff$ $\forall\varepsilon > 0\ \exists\delta > 0,\ |x-a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon$

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3. f is differentiable at a $\iff$ $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists

4. f is increasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \ \longrightarrow\ f(x_1) < f(x_2)$

5. f is decreasing on E $\iff$ $\forall x_1, x_2 \in E,\ x_1 < x_2 \ \longrightarrow\ f(x_1) > f(x_2)$

6. f is integrable on $[a, b]$ $\iff$ $\forall\varepsilon > 0\ \exists P_\varepsilon,\ U(f, P) - L(f, P) < \varepsilon$

7. Riemann sum converges to $I(f)$ $\iff$ $\forall\varepsilon > 0\ \exists P_\varepsilon \subseteq \{x_0, x_1, ..., x_n\} \longrightarrow \left|\sum_{i=1}^{n} f(t_j)\Delta x_j - I(f)\right| < \varepsilon$

8. **Cauchy Criterion**: $\sum\limits_{k=1}^{\infty} a_k$ converges $\iff$ $\forall\varepsilon > 0\ \exists N \in \mathbb{N}, m > n \geq N \longrightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

1. **(10 marks)** Use definition to prove that

$$f(x) = (x^2 - 1) + (x + 1)^2$$

is continuous at $x = -1$.

2. **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = x - f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

3. **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\frac{1}{x^2} \leq \sqrt{x} \quad \text{for all} \quad x \geq 1.$$

4. **(10 marks)** Define $f(x) = x - e^{-2x}$ where $x \in \mathbb{R}$.

   Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

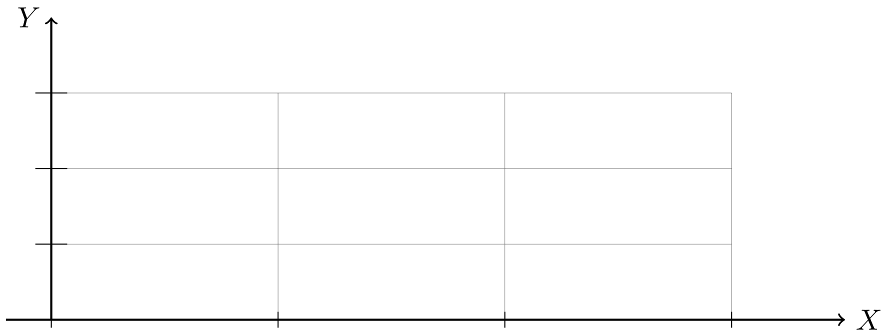
   Q2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

   Q3 **(3 marks)** Compute $(f^{-1})'(-1)$.

5. **(10 marks)** Define

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \\ 3 & \text{if } x = 1 \end{cases}$$

Draw the graph of f on $[0,2]$ and use definition to show that f is integrable on $[0,2]$.



6. **(10 marks)** Let $f(x) = (x^2 - 1) + (x - 1)^2$ where $x \in [0, 1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\} = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., 1 \right\}$$

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Show that $\displaystyle\int_{-1}^{0} f(x)\,dx + \int_0^1 g(x)\,dx = 0.$

Hint: Use integration by part to $\displaystyle\int_{-1}^{0} f(x)\,dx$ and change variable.

8. **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty} \left[\frac{a^k}{(a+1)^k} + \frac{ak-k-1}{a^{k+1}} \right] = \frac{10}{3}.$$

Hint: Use Telescoping and Geometric Series.

9. **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

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Verify your answer.

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Math SSRU

Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Final Examination (Set C)
Semester 2/2024

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1. **(10 marks)** Use definition to prove that

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is continuous at $x = 1$.

2. **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = 2x + f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

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4. **(10 marks)** Define $f(x) = 2x - e^{-x}$ where $x \in \mathbb{R}$.

   Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

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$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \\ 3 & \text{if } x = 1 \end{cases}$$

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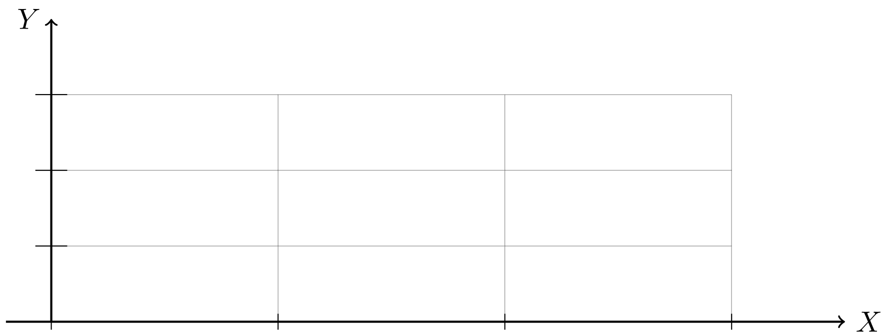

6. **(10 marks)** Let $f(x) = (x^2 - 1) + (x-1)^2$ where $x \in [0,1]$ and

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Hint: Use integration by part to $\displaystyle\int_{-1}^{0} f(x)\,dx$ and change variable.

8. **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty} \left[\frac{a^k}{(a+1)^k} + \frac{ak-k-1}{a^{k+1}} \right] = \frac{17}{4}.$$

Hint: Use Telescoping and Geometric Series.

9. **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k^3} \right)^k \text{ converges or NOT.}$$

Verify your answer.

10. **(10 marks)** Determine whether

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^4}\right)$$

is conditionally convergent or NOT.

Math SSRU

Solution Final Exam. 2/2024
MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

1. A **(10 marks)** Use definition to prove that

$$f(x) = (x^2 - 1) + (x - 1)^2$$

is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{4}\}$.
Let $x \in \mathbb{R}$ such that $|x - 1| < \delta$. Then $|x - 1| < 1$.

$$\text{So, } |x| - |1| \le |x - 1| < 1. \text{ We obtain } |x| \le 2.$$

By triangle inequality, it follows that

$$\begin{aligned} |f(x) - f(1)| &= \left|(x^2 - 1) + (x - 1)^2 - 0\right| \\ &= |(x^2 - 1) + (x^2 - 2x + 1)| = |2x^2 - 2x| \\ &= |2x(x - 1)| = 2|x||x - 1| \\ &< 2(2)\delta \\ &= 4\delta < 4 \cdot \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = 1$. □

1. B **(10 marks)** Use definition to prove that

$$f(x) = (x^2 - 1) + (x + 1)^2$$

is continuous at $x = -1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{4}\}$.
Let $x \in \mathbb{R}$ such that $|x + 1| < \delta$. Then $|x + 1| < 1$.

$$\text{So, } |x| - |1| \le |x + 1| < 1. \text{ We obtain } |x| \le 2.$$

By triangle inequality, it follows that

$$\begin{aligned} |f(x) - f(-1)| &= \left|(x^2 - 1) + (x + 1)^2 - 0\right| \\ &= |(x^2 - 1) + (x^2 + 2x + 1)| = |2x^2 + 2x| \\ &= |2x(x + 1)| = 2|x||x + 1| \\ &< 2(2)\delta \\ &= 4\delta < 4 \cdot \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = -1$. □

1. C **(10 marks)** Use definition to prove that

$$f(x) = 2(x^2 - 1) + 2(x - 1)^2$$

is continuous at $x = 1$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{8}\}$.
Let $x \in \mathbb{R}$ such that $|x - 1| < \delta$. Then $|x - 1| < 1$.

$$\text{So, } |x| - |1| \leq |x - 1| < 1. \text{ We obtain } |x| \leq 2.$$

By triangle inequility, it follows that

$$\begin{aligned} |f(x) - f(1)| &= \left|2(x^2 - 1) + 2(x - 1)^2 - 0\right| \\ &= |2(x^2 - 1) + 2(x^2 - 2x + 1)| = |4x^2 - 4x| \\ &= |4x(x + 1)| = 4|x||x - 1| \\ &< 4(2)\delta \\ &= 8\delta < 8 \cdot \frac{\varepsilon}{8} = \varepsilon. \end{aligned}$$

Therefore, f is continuous at $x = -1$. $\square$

2. A **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = x + f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

Proof. Assume that f be uniformly continuous on $\mathbb{R}$.
Let $\varepsilon > 0$. There is an $\delta_0 > 0$ such that

$$|x - a| < \delta_0 \text{ for all } x, a \in \mathbb{R} \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_0, \frac{\varepsilon}{2}\right\}$. Let $x, a \in \mathbb{R}$ such that $|x - a| < \delta$. So, $|x - a| < \delta_0$ and $|x - a| < \frac{\varepsilon}{2}$
Apply the triangle inequality and assumption, it implies that

$$\begin{aligned} |g(x) - g(a)| &= |x + f(x) - (a + f(a))| \\ &= |f(x) - f(a) + x - a| \\ &= |f(x) - f(a)| + |x - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, g is uniformly continuous on $\mathbb{R}$. □

2. B **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = x - f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

Proof. Assume that f be uniformly continuous on $\mathbb{R}$.
Let $\varepsilon > 0$. There is an $\delta_0 > 0$ such that

$$|x - a| < \delta_0 \text{ for all } x, a \in \mathbb{R} \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_0, \frac{\varepsilon}{2}\right\}$. Let $x, a \in \mathbb{R}$ such that $|x - a| < \delta$. So, $|x - a| < \delta_0$ and $|x - a| < \frac{\varepsilon}{2}$
Apply the triangle inequality and assumption, it implies that

$$\begin{aligned} |g(x) - g(a)| &= |x - f(x) - (a - f(a))| \\ &= |-(f(x) - f(a)) + x - a| \\ &\leq |-(f(x) - f(a))| + |x - a| \\ &= |f(x) - f(a)| + |x - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, g is uniformly continuous on $\mathbb{R}$. □

2. C **(10 marks)** Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$. Define

$$g(x) = 2x + f(x) \quad \text{where } x \in \mathbb{R}.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

Proof. Assume that f be uniformly continuous on $\mathbb{R}$.
Let $\varepsilon > 0$. There is an $\delta_0 > 0$ such that

$$|x - a| < \delta_0 \text{ for all } x, a \in \mathbb{R} \quad \text{implies} \quad |f(x) - f(a)| < \frac{\varepsilon}{2}.$$

Choose $\delta = \min\left\{\delta_0, \dfrac{\varepsilon}{2}\right\}$. Let $x, a \in \mathbb{R}$ such that $|x - a| < \delta$. So, $|x - a| < \delta_0$ and $|x - a| < \frac{\varepsilon}{4}$
Apply the triangle inequality and assumption, it implies that

$$\begin{aligned} |g(x) - g(a)| &= |2x + f(x) - (2a + f(a))| \\ &= |f(x) - f(a) + 2(x - a)| \\ &= |f(x) - f(a)| + 2|x - a| \\ &< \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

Thus, g is uniformly continuous on $\mathbb{R}$. □

3. A **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\frac{1}{x} \le \sqrt{x} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \frac{1}{x} - \sqrt{x} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = -\frac{1}{x^2} - \frac{1}{2\sqrt{x}}$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\frac{1}{a} - \sqrt{a} - 0 = \left(-\frac{1}{c^2} - \frac{1}{2\sqrt{c}}\right)(a-1)$$

Obviously, we see that $-\frac{1}{c^2} - \frac{1}{2\sqrt{c}} < 0$ for $c \in (1, a)$
Since $a > 1$, $a - 1 > 0$. It implies that

$$\frac{1}{a} - \sqrt{a} = \left(-\frac{1}{c^2} - \frac{1}{2\sqrt{c}}\right)(a-1) < 0$$

We conclude that $\frac{1}{x} \le \sqrt{x} \quad$ for all $\quad x \ge 1.$ □

3. B **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\frac{1}{x^2} \le \sqrt{x} \quad \text{for all} \quad x \ge 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \frac{1}{x^2} - \sqrt{x} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = -\frac{2}{x^3} - \frac{1}{2\sqrt{x}}$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a-1)$$
$$\frac{1}{a^2} - \sqrt{a} - 0 = \left(-\frac{2}{c^3} - \frac{1}{2\sqrt{c}}\right)(a-1)$$

Obviously, we see that $-\frac{2}{c^3} - \frac{1}{2\sqrt{c}} < 0$ for $c \in (1, a)$
Since $a > 1$, $a - 1 > 0$. It implies that

$$\frac{1}{a} - \sqrt{a} = \left(-\frac{2}{c^3} - \frac{1}{2\sqrt{c}}\right)(a-1) < 0$$

We conclude that $\frac{1}{x^2} \le \sqrt{x} \quad$ for all $\quad x \ge 1.$ □

3. C **(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

$$\frac{1}{x^3} \leq \sqrt{x} \quad \text{for all} \quad x \geq 1.$$

Proof. Let $a > 1$ and define

$$f(x) = \frac{1}{x^3} - \sqrt{x} \quad \text{where } x \in [1, a].$$

Then f is continuous on $[1, a]$ and differentiable on $(1, a)$. It follows that

$$f(1) = 0$$
$$f'(x) = -\frac{3}{x^4} - \frac{1}{2\sqrt{x}}$$

By the Mean Value Theorem, there is a $c \in (1, a)$ such that

$$f(a) - f(1) = f'(c)(a - 1)$$
$$\frac{1}{a^2} - \sqrt{a} - 0 = \left(-\frac{3}{c^4} - \frac{1}{2\sqrt{c}}\right)(a - 1)$$

Obviously, we see that $-\frac{3}{c^4} - \frac{1}{2\sqrt{c}} < 0$ for $c \in (1, a)$
Since $a > 1$, $a - 1 > 0$. It implies that

$$\frac{1}{a} - \sqrt{a} = \left(-\frac{3}{c^4} - \frac{1}{2\sqrt{c}}\right)(a - 1) < 0$$

We conclude that $\dfrac{1}{x^3} \leq \sqrt{x} \quad \text{for all} \quad x \geq 1.$ □

4. A **(10 marks)** Define $f(x) = x - e^{-x}$ where $x \in \mathbb{R}$.

Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x < y$. So, $-x > -y$ We obtain

$$e^{-x} > e^{-y}.$$

Thus, $-e^{-x} < -e^{-y}$. It follows that

$$\begin{aligned} x - e^{-x} &< y - e^{-y} \\ f(x) &< f(y) \end{aligned}$$

So, $f(x) \neq f(y)$. Therefore, f is injective on $\mathbb{R}$. □

Q2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

Q3 **(3 marks)** Compute $(f^{-1})'(1)$.

Solution. We see that $f'(x) = 1 + e^{-x}$ and

$$f(0) = 0 - e^0 = -1.$$

So $f^{-1}(-1) = 0$. By IFT,

$$\begin{aligned} (f^{-1})'(-1) &= \frac{1}{f'(f^{-1}(-1))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{1+e^0} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \quad \# \end{aligned}$$

4. B**(10 marks)** Define $f(x) = x - e^{-2x}$ where $x \in \mathbb{R}$.

Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x < y$. So, $-2x > -2y$ We obtain

$$e^{-2x} > e^{-2y}.$$

Thus, $-e^{-2x} < -e^{-2y}$. It follows that

$$\begin{aligned} x - e^{-2x} &< y - e^{-2y} \\ f(x) &< f(y) \end{aligned}$$

So, $f(x) \neq f(y)$. Therefore, f is injective on $\mathbb{R}$. □

Q2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

Q3 **(3 marks)** Compute $(f^{-1})'(1)$.

Solution. We see that $f'(x) = 1 + 2e^{-2x}$ and

$$f(0) = 0 - e^0 = -1.$$

So $f^{-1}(-1) = 0$. By IFT,

$$\begin{aligned} (f^{-1})'(-1) &= \frac{1}{f'(f^{-1}(-1))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{1 + 2e^0} \\ &= \frac{1}{1+2} \\ &= \frac{1}{3} \quad \# \end{aligned}$$

4. C**(10 marks)** Define $f(x) = 2x - e^{-x}$ where $x \in \mathbb{R}$.

Q1 **(5 marks)** Show that f is injective (one-to-one) on $x \in \mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ and $x \neq y$. WLOG $x < y$. So, $2x < 2y$ and $-x > -y$ We obtain

$$e^{-x} > e^{-y}.$$

Thus, $-e^{-x} < -e^{-y}$. It follows that

$$\begin{gathered} 2x - e^{-x} < 2y - e^{-y} \\ f(x) < f(y) \end{gathered}$$

So, $f(x) \neq f(y)$. Therefore, f is injective on $\mathbb{R}$. □

Q2 **(2 marks)** Use the result from 4.1 and the Inverse Function Theorem (IFT) to explain that f^{-1} differentiable on $\mathbb{R}$.

Solution. Since f is injective, f^{-1} exists. It is clear that f is continous on $\mathbb{R}$. By IFT, we conclude that f^{-1} differentiable on $\mathbb{R}$.

Q3 **(3 marks)** Compute $(f^{-1})'(1)$.

Solution. We see that $f'(x) = 2 + e^{-x}$ and

$$f(0) = 0 - e^0 = -1.$$

So $f^{-1}(-1) = 0$. By IFT,

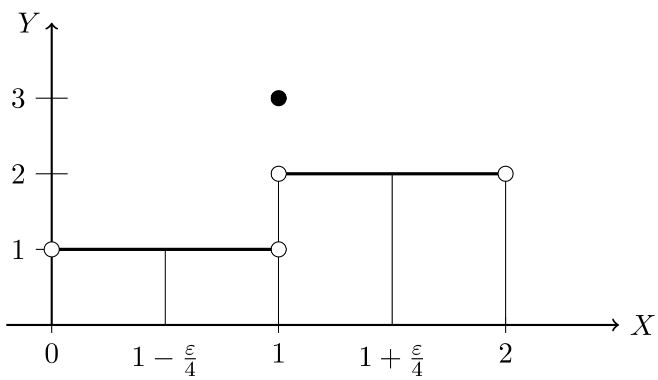
$$\begin{aligned} \left(f^{-1}\right)'(-1) &= \frac{1}{f'(f^{-1}(-1))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{2 + e^0} \\ &= \frac{1}{2+1} \\ &= \frac{1}{3} \quad \# \end{aligned}$$

5. A,B,C **(10 marks)** Define

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 2 & \text{if } x \in (1,2) \\ 3 & \text{if } x = 1 \end{cases}$$

Draw the graph of f on $[0,2]$ and use definition to show that f is integrable on $[0,2]$.

Proof. Let $\varepsilon > 0$.
Case $\varepsilon < 4$. So, $0 < \frac{\varepsilon}{4} < 1$. Choose $P = \left\{0, 1 - \dfrac{\varepsilon}{4}, 1, 1 + \dfrac{\varepsilon}{4}, 2\right\}$.



We obtain

$$\begin{aligned} U(f,P) &= 1\left(1 - \frac{\varepsilon}{4}\right) + 3\left(\frac{\varepsilon}{4}\right) + 3\left(\frac{\varepsilon}{4}\right) + 2\left(1 - \frac{\varepsilon}{4}\right) \\ L(f,P) &= 1\left(1 - \frac{\varepsilon}{4}\right) + 1\left(\frac{\varepsilon}{4}\right) + 2\left(\frac{\varepsilon}{4}\right) + 2\left(1 - \frac{\varepsilon}{4}\right) \\ U(f,P) - L(f,P) &= 2\left(\frac{\varepsilon}{4}\right) + 1\left(\frac{\varepsilon}{4}\right) = \frac{3}{4} \cdot \varepsilon < 1 \cdot \varepsilon = \varepsilon. \end{aligned}$$

Case $\varepsilon \geq 4$. Choose $P = \{0, 1, 2\}$. Then

$$\begin{aligned} U(f,P) &= 3\,(1-0) + 3\,(2-1) \\ L(f,P) &= 1\,(1-0) + 2\,(2-1) \\ U(f,P) - L(f,P) &= 2 + 1 = 3 < 4 \leq \varepsilon. \end{aligned}$$

Thus, f is integrable on $[0,2]$. □

6. A,B,C **(10 marks)** Let $f(x) = (x^2-1)+(x-1)^2$ where $x \in [0,1]$ and

$$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\} = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., 1 \right\}$$

be a partition of $[0,1]$. Find the **Riemann sum** of f and find $\boldsymbol{I(f)}$ on $[0,1]$.

Solution. Choose **The Right End Point** , i.e., $f(t_j) = f(\frac{j}{n})$ on the subinterval $[x_{j-1}, x_j]$ and

$$\Delta x_j = \frac{j}{n} - \frac{(j-1)}{n} = \frac{1}{n} \quad \text{for all } j = 1, 2, 3, ..., n.$$

From $f(x) = (x^2-1)+(x-1)^2 = 2x^2 - 2x$. We obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(t_j)\Delta x_j &= \sum_{j=1}^{n} f\left(\frac{j}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{j=1}^{n}\left[2\left(\frac{j}{n}\right)^2 - 2\left(\frac{j}{n}\right)\right] \\
&= \frac{1}{n}\left[\sum_{j=1}^{n}\frac{2j^2}{n^2} - \sum_{j=1}^{n}\frac{2j}{n}\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}\sum_{j=1}^{n} j^2 - \frac{2}{n}\sum_{j=1}^{n} j\right] \\
&= \frac{1}{n}\left[\frac{2}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\cdot\frac{n(n+1)}{2}\right] \\
&= \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{n}
\end{aligned}$$

Thus,

$$I(f) = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{n} = \frac{1}{3} - 1 = -\frac{1}{3} \quad \#$$

7. A,B,C **(10 marks)** Let g be differentiable and integrable on $\mathbb{R}$. Define

$$f(x) = \int_1^{-x} 2g(t^2)\,dt \quad \text{where } x \in \mathbb{R}.$$

Show that $\int_{-1}^0 f(x)\,dx + \int_0^1 g(x)\,dx = 0$.

Hint: Use integration by part to $\int_{-1}^0 f(x)\,dx$ and change variable.

Solution. By the First Fundamental Theorem of Calculus and Chain rule,

$$f'(x) = 2g((-x)^2) \cdot (-x)' = 2g(x^2) \cdot (-1) = -2g(x^2).$$

We have

$$f(-1) = \int_1^1 2g(t^2)\,dt = 0$$

By integration by part, we obtain

$$\begin{aligned}
\int_{-1}^0 f(x)\,dx &= \int_{-1}^0 x' f(x)\,dx = [xf(x)]_{-1}^0 - \int_{-1}^0 xf'(x)\,dx \\
&= 0f(0) - (-1)f(-1) - \int_{-1}^0 x \cdot (-2)g(x^2)\,dx \\
&= 0 - 0 + \int_{-1}^0 2x \cdot g(x^2)\,dx \\
&= \int_{-1}^0 g(x^2) \cdot (x^2)'\,dx \\
&= \int_{-1}^0 g(\phi(x))\phi'(x)\,dx && \text{Change of Variable } \phi(x) = x^2 \\
&= \int_{\phi(-1)}^{\phi(0)} g(t)\,dt \\
&= \int_1^0 g(t)\,dt \\
&= -\int_0^1 g(t)\,dt \\
&= -\int_0^1 g(x)\,dx
\end{aligned}$$

Thus, $\int_{-1}^0 f(x)\,dx + \int_0^1 g(x)\,dx = 0$.

8. A **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right]=\frac{5}{2}.$$

Hint: Use Telescoping and Geometric Series.

Solution. We consider

$$\begin{aligned}\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}} &= \left(\frac{a}{a+1}\right)^k+\frac{ak}{a^{k+1}}-\frac{k+1}{a^{k+1}}\\ &= \left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\end{aligned}$$

So, the above sequence consist of a geometric and telescoping sequences. It follows that

$$\begin{aligned}\frac{5}{2}=\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right] &= \sum_{k=1}^{\infty}\left[\left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right)\right]\\ &= \sum_{k=1}^{\infty}\left(\frac{a}{a+1}\right)^k+\sum_{k=1}^{\infty}\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\\ &= \frac{\frac{a}{a+1}}{1-\frac{a}{a+1}}+\left(\frac{1}{a}-\lim_{k\to\infty}\frac{1}{(k+1)a^{k+1}}\right) \qquad \text{if } a>1\\ &= a+\left(\frac{1}{a}-0\right)=\frac{a^2+1}{a}\end{aligned}$$

We obtain $5a=2(a^2+1)$ or $(2a-1)(a-2)=2a^2-5a+2=0$.
Then, $a=2,\frac{1}{2}$. But $a>1$. Thus, $a=2$ #

8. B **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right]=\frac{10}{3}.$$

Hint: Use Telescoping and Geometric Series.

Solution. We consider

$$\begin{aligned}\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}} &= \left(\frac{a}{a+1}\right)^k+\frac{ak}{a^{k+1}}-\frac{k+1}{a^{k+1}}\\ &= \left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\end{aligned}$$

So, the above sequence consist of a geometric and telescoping sequences. It follows that

$$\begin{aligned}\frac{10}{3}=\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right] &= \sum_{k=1}^{\infty}\left[\left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right)\right]\\ &= \sum_{k=1}^{\infty}\left(\frac{a}{a+1}\right)^k+\sum_{k=1}^{\infty}\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\\ &= \frac{\frac{a}{a+1}}{1-\frac{a}{a+1}}+\left(\frac{1}{a}-\lim_{k\to\infty}\frac{1}{(k+1)a^{k+1}}\right) \qquad \text{if } a>1\\ &= a+\left(\frac{1}{a}-0\right)=\frac{a^2+1}{a}\end{aligned}$$

We obtain $10a=3(a^2+1)$ or $(3a-1)(a-3)=3a^2-10a+3=0$.
Then, $a=3,\frac{1}{3}$. But $a>1$. Thus, $a=3$ #

8. C **(10 marks)** Find $\boldsymbol{a} \in \mathbb{R}$ satisfying

$$\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right]=\frac{17}{4}.$$

Hint: Use Telescoping and Geometric Series.

Solution. We consider

$$\begin{aligned}\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}} &= \left(\frac{a}{a+1}\right)^k+\frac{ak}{a^{k+1}}-\frac{k+1}{a^{k+1}}\\ &= \left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\end{aligned}$$

So, the above sequence consist of a geometric and telescoping sequences. It follows that

$$\begin{aligned}\frac{17}{4}=\sum_{k=1}^{\infty}\left[\frac{a^k}{(a+1)^k}+\frac{ak-k-1}{a^{k+1}}\right] &= \sum_{k=1}^{\infty}\left[\left(\frac{a}{a+1}\right)^k+\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right)\right]\\ &= \sum_{k=1}^{\infty}\left(\frac{a}{a+1}\right)^k+\sum_{k=1}^{\infty}\left(\frac{k}{a^k}-\frac{k+1}{a^{k+1}}\right).\\ &= \frac{\frac{a}{a+1}}{1-\frac{a}{a+1}}+\left(\frac{1}{a}-\lim_{k\to\infty}\frac{1}{(k+1)a^{k+1}}\right) \qquad \text{if } a>1\\ &= a+\left(\frac{1}{a}-0\right)=\frac{a^2+1}{a}\end{aligned}$$

We obtain $17a=4(a^2+1)$ or $(4a-1)(a-4)=4a^2-17a+4=0$.
Then, $a=4,\frac{1}{4}$. But $a>1$. Thus, $a=4$ #

9. A **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k} \right)^k \text{ converges or NOT.}$$

Verify your answer.

Solution.

Proof. Use the Root Test, we consider

$$\begin{aligned} r &= \limsup_{k\to\infty} \left| \left(\frac{a + (-1)^k}{k} \right)^k \right|^{\frac{1}{k}} \\ &= \limsup_{k\to\infty} \left| \frac{a + (-1)^k}{k} \right| \\ &= \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k}, \frac{|a+1|}{k} \right\} \end{aligned}$$

Whatever we obtain $\sup \left\{ \frac{|a-1|}{k}, \frac{|a+1|}{k} \right\} = \frac{|a-1|}{k}$ or $\frac{|a+1|}{k}$. For any $a \in \mathbb{R}$, it's clear that

$$r = \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k}, \frac{|a+1|}{k} \right\} = \lim_{n\to\infty} \frac{|a-1|}{k} = 0 < 1$$

$$r = \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k}, \frac{|a+1|}{k} \right\} = \lim_{n\to\infty} \frac{|a+1|}{k} = 0 < 1$$

We conclude that $\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k} \right)^k$ converges absolutely for all $a \in \mathbb{R}$. $\square$

9. B **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty}\left(\frac{a+(-1)^k}{k^2}\right)^k \text{ converges or NOT.}$$

Verify your answer.

Solution.

Proof. Use the Root Test, we consider

$$\begin{aligned} r &= \limsup_{k\to\infty}\left|\left(\frac{a+(-1)^k}{k^2}\right)^k\right|^{\frac{1}{k}} \\ &= \limsup_{k\to\infty}\left|\frac{a+(-1)^k}{k^2}\right| \\ &= \lim_{n\to\infty}\sup\left\{\frac{|a-1|}{k^2},\frac{|a+1|}{k^2}\right\} \end{aligned}$$

Whatever we obtain $\sup\left\{\frac{|a-1|}{k^2},\frac{|a+1|}{k^2}\right\} = \frac{|a-1|}{k^2}$ or $\frac{|a+1|}{k^2}$. For any $a \in \mathbb{R}$, it's clear that

$$r = \lim_{n\to\infty}\sup\left\{\frac{|a-1|}{k^2},\frac{|a+1|}{k^2}\right\} = \lim_{n\to\infty}\frac{|a-1|}{k^2} = 0 < 1$$
$$r = \lim_{n\to\infty}\sup\left\{\frac{|a-1|}{k^2},\frac{|a+1|}{k^2}\right\} = \lim_{n\to\infty}\frac{|a+1|}{k^2} = 0 < 1$$

We conclude that $\sum_{k=1}^{\infty}\left(\frac{a+(-1)^k}{k^2}\right)^k$ converges absolutely for all $a \in \mathbb{R}$. □

9. C **(10 marks)** Let $a \in \mathbb{R}$. Determine whether

$$\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k^3} \right)^k \text{ converges or NOT.}$$

Verify your answer.

Solution.

Proof. Use the Root Test, we consider

$$\begin{aligned} r &= \limsup_{k\to\infty} \left| \left(\frac{a + (-1)^k}{k^3} \right)^k \right|^{\frac{1}{k}} \\ &= \limsup_{k\to\infty} \left| \frac{a + (-1)^k}{k^3} \right| \\ &= \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k^3}, \frac{|a+1|}{k^3} \right\} \end{aligned}$$

Whatever we obtain $\sup \left\{ \frac{|a-1|}{k^3}, \frac{|a+1|}{k^3} \right\} = \frac{|a-1|}{k^3}$ or $\frac{|a+1|}{k^3}$. For any $a \in \mathbb{R}$, it's clear that

$$r = \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k^3}, \frac{|a+1|}{k^3} \right\} = \lim_{n\to\infty} \frac{|a-1|}{k^3} = 0 < 1$$
$$r = \lim_{n\to\infty} \sup \left\{ \frac{|a-1|}{k^3}, \frac{|a+1|}{k^3} \right\} = \lim_{n\to\infty} \frac{|a+1|}{k^3} = 0 < 1$$

We conclude that $\sum_{k=1}^{\infty} \left(\frac{a + (-1)^k}{k^3} \right)^k$ converges absolutely for all $a \in \mathbb{R}$. □

10. A **(10 marks)** Determine whether

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^2}\right)$$

is conditionally convergent or NOT.

Solution. We consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \ln\left(1+\frac{1}{k^2}\right)\right| = \sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right).$$

Use the limit comparison test by $b_k = \frac{1}{k^2}$,

$$\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k^2}\right)}{\frac{1}{k^2}} = \lim_{k\to\infty}\frac{\frac{1}{1+\frac{1}{k^2}}\cdot\left(-\frac{2}{k^3}\right)}{-\frac{2}{k^3}} = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k^2}}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^2}$ converges ($p = 2$), by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right) \qquad \text{converges.}$$

Thus,

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^2}\right) \qquad \text{is absolutely convergent.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^2}\right) \qquad \text{is not conditionally convergent.}$$

10. B **(10 marks)** Determine whether

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^3}\right)$$

is conditionally convergent or NOT.

Solution. We consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \ln\left(1+\frac{1}{k^3}\right)\right| = \sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^3}\right).$$

Use the limit comparison test by $b_k = \frac{1}{k^3}$,

$$\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k^3}\right)}{\frac{1}{k^3}} = \lim_{k\to\infty}\frac{\frac{1}{1+\frac{1}{k^3}}\cdot\left(-\frac{3}{k^4}\right)}{-\frac{3}{k^4}} = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k^3}}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^3}$ converges ($p = 3$), by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^3}\right) \qquad \text{converges.}$$

Thus,

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^3}\right) \qquad \text{is absolutely convergent.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^3}\right) \qquad \text{is not conditionally convergent.}$$

10. C **(10 marks)** Determine whether

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^4}\right)$$

is conditionally convergent or NOT.

Solution. We consider

$$\sum_{k=1}^{\infty}\left|(-1)^k \ln\left(1+\frac{1}{k^4}\right)\right| = \sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^4}\right).$$

Use the limit comparison test by $b_k = \frac{1}{k^4}$,

$$\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k^4}\right)}{\frac{1}{k^4}} = \lim_{k\to\infty}\frac{\frac{1}{1+\frac{1}{k^4}}\cdot\left(-\frac{4}{k^5}\right)}{-\frac{4}{k^5}} = \lim_{k\to\infty}\left(\frac{1}{1+\frac{1}{k^4}}\right) = 1 > 0$$

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^4}$ converges ($p = 4$), by the Limit Comparision Test, it implies that

$$\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^4}\right) \qquad \text{converges.}$$

Thus,

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^4}\right) \qquad \text{is absolutely convergent.}$$

Therefore, we conclude that

$$\sum_{k=1}^{\infty}(-1)^k \ln\left(1+\frac{1}{k^4}\right) \qquad \text{is not conditionally convergent.}$$