Suan Sunandha Rajabhat University
Faculty of Education, Division of Mathematics
Midterm Examination
Semester 2/2023

Course ID	**Course Name**	**Test Time**	**Full Scores**
MAC3309	Mathematical Analysis	5pm - 8pm Mon 29 Feb 2024	100 marks 25%

Name.. **ID**.................................... **Section**...............

Direction

1. 10 questions of all 12 pages.
2. Write obviously your name, id and section all pages.
3. Don't take text books and others come to the test room.
4. Cannot answer sheets out of test room.
5. Deliver to the staff if you make a mistake in the test room.

Your signature

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Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Scores											

Some Definition to prove this examination.

1. $\lim\limits_{n\to\infty} x_n = a \iff \forall\varepsilon > 0\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow |x_n - a| < \varepsilon$

2. $\lim\limits_{n\to\infty} x_n = +\infty \iff \forall M \in \mathbb{R}\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow x_n > M$

3. $\lim\limits_{n\to\infty} x_n = -\infty \iff \forall M \in \mathbb{R}\ \exists N \in \mathbb{N},\ n \geq N \longrightarrow x_n < M$

4. $\lim\limits_{x\to a} f(x) = L \iff \forall\varepsilon > 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow |f(x) - L| < \varepsilon$

5. $\lim\limits_{x\to\infty} f(x) = L \iff \forall\varepsilon > 0\ \exists M \in \mathbb{R},\ x > M \longrightarrow |f(x) - L| < \varepsilon$

6. $\lim\limits_{x\to-\infty} f(x) = L \iff \forall\varepsilon > 0\ \exists M \in \mathbb{R},\ x < M \longrightarrow |f(x) - L| < \varepsilon$

7. $\lim\limits_{x\to a} f(x) = +\infty \iff \forall M > 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow f(x) > M$

8. $\lim\limits_{x\to a} f(x) = -\infty \iff \forall M < 0\ \exists\delta > 0,\ 0 < |x - a| < \delta \longrightarrow f(x) < M$

1. **(10 marks)** Let a and b be real numbers. Prove that

$$a^2 + b^2 + \frac{1}{2} \geq a + b.$$

2. **(10 marks)** Let x and y be real numbers. Prove that

$$\text{if} \quad |2x - y| = |x - 2y|, \quad \text{then} \quad |x + y| \leq 2|x|.$$

3. **(10 marks)** Define the set

$$A = \left\{ 6 + \frac{7}{n^2} : n \in \mathbb{N} \right\}.$$

Find **sup *A*** and **inf *A*** with proving them.

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{2n^2+4}{6n^2+7} = \frac{1}{3}.$$

5. **(10 marks)** Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Prove that

if x_n and $x_n + y_n$ coverges, then y_n also converges.

6. **(10 marks)** Use definition to prove that

$$\left\{\frac{1}{n(n+1)}\right\} \text{ is a Caucy sequence.}$$

7. **(10 marks)** Let A and B be non-empty subset of $\mathbb{R}$.
   Assume that A is open and B is closed.

   Determine whether $A - B$ is open. Verify your answer.

8. **(10 marks)** Use definition to prove that

$$\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}.$$

9. **(10 marks)** Use definition prove that

$$\lim_{x\to 1^+} \frac{x}{1-x^2} = -\infty.$$

10. **(10 marks)** Let f and g be functions on $\mathbb{R}$ such that

$$g(x) > 1 \quad \text{for all } x \in \mathbb{R}$$

Let a be a limit point of $\mathbb{R}$ and $f(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that if $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} \frac{g(x)}{|f(x)|} = +\infty.$$

Solution Midterm Exam. 2/2023
MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

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1. **(10 marks)** Let a and b be real numbers. Prove that

$$a^2 + b^2 + \frac{1}{2} \geq a + b.$$

TYPE I

Proof. Let a and b be real numbers. By the fact that

$$(a + b - 1)^2 \geq 0 \text{ and } (a - b)^2 \geq 0.$$

We obtain

$$\begin{aligned}
(a + b - 1)^2 + (a - b)^2 &\geq 0 \\
(a + b)^2 - 2(a + b) + 1 + a^2 - 2ab + b^2 &\geq 0 \\
a^2 + 2ab + b^2 - 2a - 2b + 1 + a^2 - 2ab + b^2 &\geq 0 \\
2a^2 + 2b^2 - 2a - 2b + 1 &\geq 0 \\
2(a^2 + b^2) + 1 &\geq 2(a + b) \\
a^2 + b^2 + \frac{1}{2} &\geq a + b
\end{aligned}$$

□

TYPE II

Proof. Let a and b be real numbers. By the fact that

$$(a - \tfrac{1}{2})^2 \geq 0 \text{ and } (b - \tfrac{1}{2})^2 \geq 0.$$

We obtain

$$\begin{aligned}
\left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 &\geq 0 \\
a^2 - a + \frac{1}{4} + b^2 - b + \frac{1}{4} &\geq 0 \\
a^2 + b^2 + \frac{1}{2} &\geq a + b
\end{aligned}$$

□

2. **(10 marks)** Let x and y be real numbers. Prove that

$$\text{if} \quad |2x-y| = |x-2y|, \quad \text{then} \quad |x+y| \leq 2|x|.$$

Proof. Let x be a real numbers.
Assume that $|2x-y| = |x-2y|$. Then

$$\begin{aligned} |2x-y|^2 &= |x-2y|^2 \\ (2x-y)^2 &= (x-2y)^2 \\ 4x^2-4xy+y^2 &= x^2-4xy+4y^2 \\ 3x^2 &= 3y^2 \\ x^2 &= y^2 \\ \sqrt{x^2} &= \sqrt{y^2} \\ |x| &= |y| \end{aligned}$$

From Triangle inequality, we obtain

$$|x+y| \leq |x|+|y| = |x|+|x| = 2|x|$$

□

3. **(10 marks)** Define the set

$$A = \left\{ 6 + \frac{7}{n^2} : n \in \mathbb{N} \right\}.$$

Find **sup *A*** and **inf *A*** with proving them.

Claim that $\inf A = 6$ and $\sup A = 13$

Proof. $\inf A = 6$

Let $n \in \mathbb{N}$. Then $n > 0$. So, $\frac{7}{n^2} > 0$. It's clear that

$$6 \leq 6 + \frac{7}{n^2}.$$

Thus, 6 is a lower bound of A.

Suppose that there is a lower bound ℓ_0 of A such that $\ell_0 > 6$. It follows that

$$\ell_0 \leq 6 + \frac{7}{n^2} \quad \text{for all } n \in \mathbb{N}. \qquad (*)$$

From $\dfrac{\ell_0 - 6}{7} > 0$, by Archimendean property, there is an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\ell_0 - 6}{7}.$$

Since $n_0 \geq 1$, $n_0^2 \geq n_0$. We obtain

$$\frac{7}{n_0^2} \leq \frac{7}{n_0} < \ell_0 - 6.$$

So, $6 + \dfrac{7}{n_0^2} < \ell_0$. This is contradiction to $(*)$. Therefore, $\inf A = 6$.

$\sup A = 13$

Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $n^2 \geq 1$. We obtain

$$\begin{aligned} \frac{1}{n^2} &\leq 1 \\ \frac{7}{n^2} &\leq 7 \\ 6 + \frac{7}{n^2} &\leq 13 \end{aligned}$$

Thus, 13 is an upper bound of A.

Let u be an upper bound of A. Then

$$6 + \frac{7}{n^2} \leq u \quad \text{for all } n \in \mathbb{N}.$$

Choose $n = 1$, we obtain

$$13 \leq u$$

Therefore, $\sup A = 13$. □

4. **(10 marks)** Use Definition to prove that

$$\lim_{n\to\infty} \frac{2n^2+4}{6n^2+7} = \frac{1}{3}.$$

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{5}$.
Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{1}{n} \leq \frac{1}{N}$. Since $18n^2 + 21 > 18n^2$, $\frac{1}{18n^2+21} < \frac{1}{18n^2}$.
From $n^2 \geq n$, we have $\frac{1}{n^2} < \frac{1}{n}$. It follows that

$$\begin{aligned}\left|\frac{2n^2+4}{6n^2+7} - \frac{1}{3}\right| &= \left|\frac{3(2n^2+4)-(6n^2+7)}{3(6n^2+7)}\right| = \left|\frac{6n^2+12-6n^2-7}{18n^2+21}\right| \\ &= \left|\frac{5}{18n^2+21}\right| \\ &= \frac{5}{18n^2+21} \leq \frac{5}{18n^2} \leq \frac{5}{n^2} \leq \frac{5}{n} \leq \frac{5}{N} < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{2n^2+4}{6n^2+7} = \frac{1}{3}$. □

5. **(10 marks)** Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Prove that

if x_n and $x_n + y_n$ coverges, then y_n also converges.

Proof. Assume that $x_n \to L$ and $x_n + y_n \to M$ as $n \to \infty$.
We will to prove that $y_n \to M - L$ as $n \to \infty$.
Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \text{implies} \quad |x_n - L| < \frac{\varepsilon}{2}$$

and

$$n \geq N_2 \quad \text{implies} \quad |(x_n + y_n) - M| < \frac{\varepsilon}{2}$$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain

$$\begin{aligned}|y_n - (M-L)| &= |((x_n+y_n) - M) - (x_n - L)| \\ &\leq |(x_n+y_n) - M| + |x_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, y_n converges. □

6. **(10 marks)** Use definition to prove that

$$\left\{\frac{1}{n(n+1)}\right\} \text{ is a Caucy sequence.}$$

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{2}$.
Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\dfrac{1}{n} \leq \dfrac{1}{N}$ and $\dfrac{1}{m} \leq \dfrac{1}{N}$.
From $n^2 > 0$ and $m^2 > 0$, we have

$$n(n+1) = n^2 + n > n \quad \text{and} \quad m(m+1) = m^2 + m > m.$$

So,

$$\frac{1}{n(n+1)} < \frac{1}{n} \quad \text{and} \quad \frac{1}{m(m+1)} \leq \frac{1}{m}.$$

It follows that

$$\begin{aligned}
\left|\frac{1}{n(n+1)} - \frac{1}{m(m+1)}\right| &= \left|\frac{1}{n(n+1)}\right| + \left|\frac{1}{m(m+1)}\right| \\
&= \frac{1}{n(n+1)} + \frac{1}{m(m+1)} \\
&\leq \frac{1}{n} + \frac{1}{m} \\
&\leq \frac{1}{N} + \frac{1}{N} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, $\left\{\dfrac{1}{n(n+1)}\right\}$ is Cauchy.

□

7. **(10 marks)** Let A and B be non-empty subset of $\mathbb{R}$.
Assume that A is open and B is closed.

Determine whether $A - B$ is open. Verify your answer.

Answer : $A - B$ is open.

Proof. Assume that A is open and B is closed. Then B^c is open.
By theorem, it implies that $A \cap B^c$ is open. Therefore,

$$A - B = A \cap B^c \text{ is open.}$$

□

8. **(10 marks)** Use definition to prove that

$$\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \varepsilon\right\}$. Suppose that $0 < |x-1| < \delta$. Then $0 < |x-1| < \frac{1}{2}$. We have

$$\begin{aligned}
-\frac{1}{2} &< x-1 < \frac{1}{2} \\
\frac{1}{2} &< x < \frac{3}{2} \\
\frac{1}{4} &< x^2 < \frac{9}{4} \\
\frac{5}{4} &< x^2+1 < \frac{13}{4}
\end{aligned}$$

It follows that

$$\frac{3}{2} < x+1 < \frac{5}{2} \quad \text{and} \quad \frac{5}{2} < 2(x^2+1) < \frac{13}{2}$$

So, $|x+1| < \dfrac{5}{2}$ and $\dfrac{1}{2(x^2+1)} < \dfrac{2}{5}$ Then,

$$\begin{aligned}
\left|\frac{1}{x^2+1} - \frac{1}{2}\right| &= \left|\frac{2-(x^2+1)}{2(x^2+1)}\right| = \left|\frac{1-x^2}{2(x^2+1)}\right| \\
&= \frac{|x^2-1|}{2(x^2+1)} = \frac{|(x-1)(x+1)|}{2(x^2+1)} \\
&= |x-1| \cdot |x+1| \cdot \frac{1}{2(x^2+1)} \\
&< \delta \cdot \frac{5}{2} \cdot \frac{2}{5} = \delta < \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x\to 1} \dfrac{1}{x^2+1} = \dfrac{1}{2}$. □

9. **(10 marks)** Use definition prove that

$$\lim_{x\to 1^+} \frac{x}{1-x^2} = -\infty.$$

Proof. Let $M < 0$. Choose $\delta = \min\left\{1, -\frac{1}{3M}\right\}$. Then $0 < \delta \le 1$ and $0 < \delta \le -\frac{1}{3M}$.
It is equivalent to

$$-\frac{1}{3\delta} \le M.$$

Let $x \in \mathbb{R}$ such that $0 < x - 1 < \delta$. Then $0 < x - 1 < 1$ or $1 < x < 2$. So, $2 < x + 1 < 3$. We obatin

$$\frac{1}{x-1} > \frac{1}{\delta} \quad \text{and} \quad \frac{1}{x+1} > \frac{1}{3}$$

It is clear that $x - 1 > 0$ and $x + 1 > 0$. Then

$$\begin{aligned} x &> 1 \\ x \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} &> 1 \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} > \frac{1}{\delta} \cdot \frac{1}{3} \\ -x \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} &< -\frac{1}{\delta} \cdot \frac{1}{3} \\ \frac{x}{1-x^2} &< -\frac{1}{3\delta} \le M. \end{aligned}$$

Thus, $\lim_{x\to 1^+} \frac{x}{1-x^2} = -\infty$. □

10. **(10 marks)** Let f and g be functions on $\mathbb{R}$ such that

$$g(x) > 1 \quad \text{for all } x \in \mathbb{R}$$

Let a be a limit point of $\mathbb{R}$ and $f(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that if $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} \frac{g(x)}{|f(x)|} = +\infty.$$

Proof. Assume that f and g are functions on $\mathbb{R}$ such that

$$g(x) > 1 \quad \text{for all } x \in \mathbb{R}$$

Let a be a limit point of $\mathbb{R}$ and $f(x) \neq 0$ for all $x \in \mathbb{R}$.
Suppose that $f(x) \to 0$ as $x \to a$.
Let $M > 0$. There is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x)| < \frac{1}{M}.$$

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$. We obtain $|f(x)| > 0$ and $\dfrac{1}{|f(x)|} > M$. From $g(x) \geq 1$, it follows that

$$\frac{g(x)}{|f(x)|} \geq \frac{1}{|f(x)|} > M.$$

Thus, $\dfrac{g(x)}{|f(x)|} \to +\infty$ as $x \to a$. □