

Suan Sunandha Rajabhat University Faculty of Education, Division of Mathematics Midterm Examination Semester 2/2023

Course ID	Course Name	Test Time	Full Scores 100 marks		
MAC3309	Mathematical	5pm - 8pm			
	Analysis	Mon 29 Feb 2024	25%		
2.7			G		
Name		ID	\dots Section		

Direction

- 1. 10 questions of all 12 pages.
- 2. Write obviously your name, id and section all pages.
- 3. Don't take text books and others come to the test room.
- 4. Cannot answer sheets out of test room.
- 5. Deliver to the staff if you make a mistake in the test room.

Your signature

Lecturer: Assistant Professor Thanatyod Jampawai, Ph.D.

No.	1	2	3	4	5	6	7	8	9	10	Total
Scores											

Some Definition to prove this examination.

1.
$$\lim_{n \to \infty} x_n = a$$
 $\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ n \ge N \longrightarrow |x_n - a| < \varepsilon$

2.
$$\lim_{n\to\infty} x_n = +\infty$$
 \iff $\forall M \in \mathbb{R} \ \exists N \in \mathbb{N}, \ n \geq N \longrightarrow x_n > M$

3.
$$\lim_{n \to \infty} x_n = -\infty \iff \forall M \in \mathbb{R} \ \exists N \in \mathbb{N}, \ n \ge N \longrightarrow x_n < M$$

$$4. \quad \lim_{x\to a} f(x) = L \qquad \Longleftrightarrow \quad \forall \varepsilon > 0 \ \exists \delta > 0, \ 0 < |x-a| < \delta \longrightarrow |f(x)-L| < \varepsilon$$

5.
$$\lim_{x \to \infty} f(x) = L \iff \forall \varepsilon > 0 \ \exists M \in \mathbb{R}, \ x > M \longrightarrow |f(x) - L| < \varepsilon$$

6.
$$\lim_{x \to -\infty} f(x) = L \iff \forall \varepsilon > 0 \ \exists M \in \mathbb{R}, \ x < M \longrightarrow |f(x) - L| < \varepsilon$$

7.
$$\lim_{x \to a} f(x) = +\infty \iff \forall M > 0 \ \exists \delta > 0, \ 0 < |x - a| < \delta \longrightarrow f(x) > M$$

8.
$$\lim_{x \to a} f(x) = -\infty \iff \forall M < 0 \ \exists \delta > 0, \ 0 < |x - a| < \delta \longrightarrow f(x) < M$$



1. **(10 marks)** Let a and b be real numbers. Prove that

$$a^2 + b^2 + \frac{1}{2} \ge a + b.$$



2. (10 marks) Let x and y be real numbers. Prove that

$$\text{if} \quad |2x-y| = |x-2y|, \quad \text{ then } \quad |x+y| \leq 2|x|.$$

3. (10 marks) Define the set

$$A = \left\{ 6 + \frac{7}{n^2} : n \in \mathbb{N} \right\}.$$

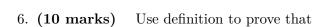
Find $\sup A$ and $\inf A$ with proving them.



$$\lim_{n \to \infty} \frac{2n^2 + 4}{6n^2 + 7} = \frac{1}{3}.$$

5. (10 marks) Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Prove that

if x_n and $x_n + y_n$ coverges, then y_n also converges.

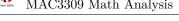


$$\left\{\frac{1}{n(n+1)}\right\}$$
 is a Caucy sequence.

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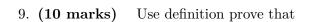
7. (10 marks) Let A and B be non-empty subset of \mathbb{R} . Assume that A is open and B is closed.

Determine whether A - B is open. Verify your answer.



8. (10 marks) Use definition to prove that

$$\lim_{x \to 1} \frac{1}{x^2 + 1} = \frac{1}{2}.$$



$$\lim_{x\to 1^+}\frac{x}{1-x^2}=-\infty.$$

10. (10 marks) Let f and g be functions on \mathbb{R} such that

$$g(x) > 1$$
 for all $x \in \mathbb{R}$

Let a be a limit point of $\mathbb R$ and $f(x) \neq 0$ for all $x \in \mathbb R$. Prove that if $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} \frac{g(x)}{|f(x)|} = +\infty.$$



Solution Midterm Exam. 2/2023 MAC3309 Mathematical Analysis

Created by Assistant Professor Thanatyod Jampawai, Ph.D.

1. (10 marks) Let a and b be real numbers. Prove that

$$a^2 + b^2 + \frac{1}{2} \ge a + b.$$

TYPE I

Proof. Let a and b be real numbers. By the fact that

$$(a+b-1)^2 \ge 0$$
 and $(a-b)^2 \ge 0$.

We obtain

$$(a+b-1)^{2} + (a-b)^{2} \ge 0$$

$$(a+b)^{2} - 2(a+b) + 1 + a^{2} - 2ab + b^{2} \ge 0$$

$$a^{2} + 2ab + b^{2} - 2a - 2b + 1 + a^{2} - 2ab + b^{2} \ge 0$$

$$2a^{2} + 2b^{2} - 2a - 2b + 1 \ge 0$$

$$2(a^{2} + b^{2}) + 1 \ge 2(a+b)$$

$$a^{2} + b^{2} + \frac{1}{2} \ge a + b$$

TYPE II

Proof. Let a and b be real numbers. By the fact that

$$(a - \frac{1}{2})^2 \ge 0$$
 and $(b - \frac{1}{2})^2 \ge 0$.

We obtain

$$\left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 \ge 0$$

$$a^2 - a + \frac{1}{4} + b^2 - b + \frac{1}{4} \ge 0$$

$$a^2 + b^2 + \frac{1}{2} \ge a + b$$

2. (10 marks) Let x and y be real numbers. Prove that

if
$$|2x - y| = |x - 2y|$$
, then $|x + y| \le 2|x|$.

Proof. Let x be a real numbers.

Assume that |2x - y| = |x - 2y|. Then

$$|2x - y|^{2} = |x - 2y|^{2}$$

$$(2x - y)^{2} = (x - 2y)^{2}$$

$$4x^{2} - 4xy + y^{2} = x^{2} - 4xy + 4y^{2}$$

$$3x^{2} = 3y^{2}$$

$$x^{2} = y^{2}$$

$$\sqrt{x^{2}} = \sqrt{y^{2}}$$

$$|x| = |y|$$

From Triangle inequality, we obtain

$$|x + y| \le |x| + |y| = |x| + |x| = 2|x|$$

3. (10 marks) Define the set

$$A = \left\{ 6 + \frac{7}{n^2} : n \in \mathbb{N} \right\}.$$

Find $\sup A$ and $\inf A$ with proving them.

Claim that $\inf A = 6$ and $\sup A = 13$

Proof. $\inf A = 6$

Let $n \in \mathbb{N}$. Then n > 0. So, $\frac{7}{n^2} > 0$. It's clear that

$$6 \le 6 + \frac{7}{n^2}.$$

Thus, 6 is a lower bound of A.

Suppose that there is a lower bound ℓ_0 of A such that $\ell_0 > 6$. It follows that

$$\ell_0 \le 6 + \frac{7}{n^2}$$
 for all $n \in \mathbb{N}$. (*)

From $\frac{\ell_0-6}{7}>0$, by Archimendean property, there is an $n_0\in\mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\ell_0 - 6}{7}.$$

Since $n_0 \ge 1$, $n_0^2 \ge n_0$. We obtain

$$\frac{7}{n_0^2} \le \frac{7}{n_0} < \ell_0 - 6.$$

So, $6 + \frac{7}{n_0^2} < \ell_0$. This is contradiction to (*). Therefore, $\inf A = 6$.

$\sup A = 13$

Let $n \in \mathbb{N}$. Then $n \geq 1$. So, $n^2 \geq 1$. We obtain

$$\frac{1}{n^2} \le 1$$

$$\frac{7}{n^2} \le 7$$

$$6 + \frac{7}{n^2} \le 13$$

Thus, 13 is an upper bound of A.

Let u be an upper bound of A. Then

$$6 + \frac{7}{n^2} \le u$$
 for all $n \in \mathbb{N}$.

Choose n = 1, we obtain

$$13 \le u$$

Therefore, $\sup A = 13$.

4. (10 marks) Use Definition to prove that

$$\lim_{n \to \infty} \frac{2n^2 + 4}{6n^2 + 7} = \frac{1}{3}.$$

Proof. Let $\varepsilon > 0$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{5}$. Let $n \in \mathbb{N}$ such that $n \geq N$. We obtain $\frac{1}{n} \leq \frac{1}{N}$. Since $18n^2 + 21 > 18n^2$, $\frac{1}{18n^2 + 21} < \frac{1}{18n^2}$. From $n^2 \geq n$, we have $\frac{1}{n^2} < \frac{1}{n}$. It follows that

$$\begin{split} \left| \frac{2n^2 + 4}{6n^2 + 7} - \frac{1}{3} \right| &= \left| \frac{3(2n^2 + 4) - (6n^2 + 7)}{3(6n^2 + 7)} \right| = \left| \frac{6n^2 + 12 - 6n^2 - 7}{18n^2 + 21} \right| \\ &= \left| \frac{5}{18n^2 + 21} \right| \\ &= \frac{5}{18n^2 + 21} \le \frac{5}{18n^2} \le \frac{5}{n^2} \le \frac{5}{n} \le \frac{5}{N} < \varepsilon. \end{split}$$

Thus, $\lim_{n \to \infty} \frac{2n^2 + 4}{6n^2 + 7} = \frac{1}{3}$.

5. (10 marks) Let $\{x_n\}$ and $\{y_n\}$ be sequences in real. Prove that

if x_n and $x_n + y_n$ coverges, then y_n also converges.

Proof. Assume that $x_n \to L$ and $x_n + y_n \to M$ as $n \to \infty$.

We will to prove that $y_n \to M - L$ as $n \to \infty$.

Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that

$$n \ge N_1$$
 implies $|x_n - L| < \frac{\varepsilon}{2}$

and

$$n \ge N_2$$
 implies $|(x_n + y_n) - M| < \frac{\varepsilon}{2}$

Let $n \in \mathbb{N}$. Choose $N = \max\{N_1, N_2\}$. For each $n \geq N$, we obtain

$$|y_n - (M - L)| = |((x_n + y_n) - M) - (x_n - L)|$$

$$\leq |(x_n + y_n) - M| + |x_n - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, y_n converges.

6. (10 marks) Use definition to prove that

$$\left\{\frac{1}{n(n+1)}\right\}$$
 is a Caucy sequence.

Proof. Let $\varepsilon > 0$. By Arichimedean property, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Let $n, m \in \mathbb{N}$ such that $n, m \geq N$. Then $\frac{1}{n} \leq \frac{1}{N}$ and $\frac{1}{m} \leq \frac{1}{N}$. From $n^2 > 0$ and $m^2 > 0$, we have

$$n(n+1) = n^2 + n > n$$
 and $m(m+1) = m^2 + m > m$.

So,

$$\frac{1}{n(n+1)} < \frac{1}{n} \quad \text{and} \quad \frac{1}{m(m+1)} \le \frac{1}{m}.$$

It follows that

$$\left| \frac{1}{n(n+1)} - \frac{1}{m(m+1)} \right| = \left| \frac{1}{n(n+1)} \right| + \left| \frac{1}{m(m+1)} \right|$$

$$= \frac{1}{n(n+1)} + \frac{1}{m(m+1)}$$

$$\leq \frac{1}{n} + \frac{1}{m}$$

$$\leq \frac{1}{N} + \frac{1}{N}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\left\{\frac{1}{n(n+1)}\right\}$ is Cauchy.

7. (10 marks) Let A and B be non-empty subset of \mathbb{R} . Assume that A is open and B is closed.

Determine whether A - B is open. Verify your answer.

Answer : A - B is open.

Proof. Assume that A is open and B is closed. Then B^c is open. By theorem, it implies that $A \cap B^c$ is open. Therefore,

$$A - B = A \cap B^c$$
 is open.

8. (10 marks) Use definition to prove that

$$\lim_{x \to 1} \frac{1}{x^2 + 1} = \frac{1}{2}.$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \varepsilon\right\}$. Suppose that $0 < |x - 1| < \delta$. Then $0 < |x - 1| < \frac{1}{2}$. We have

$$\begin{aligned} -\frac{1}{2} &< x - 1 < \frac{1}{2} \\ \frac{1}{2} &< x < \frac{3}{2} \\ \frac{1}{4} &< x^2 < \frac{9}{4} \\ \frac{5}{4} &< x^2 + 1 < \frac{13}{4} \end{aligned}$$

It follows that

$$\frac{3}{2} < x + 1 < \frac{5}{2}$$
 and $\frac{5}{2} < 2(x^2 + 1) < \frac{13}{2}$

So, $|x+1| < \frac{5}{2}$ and $\frac{1}{2(x^2+1)} < \frac{2}{5}$ Then,

$$\left| \frac{1}{x^2 + 1} - \frac{1}{2} \right| = \left| \frac{2 - (x^2 + 1)}{2(x^2 + 1)} \right| = \left| \frac{1 - x^2}{2(x^2 + 1)} \right|$$

$$= \frac{|x^2 - 1|}{2(x^2 + 1)} = \frac{|(x - 1)(x + 1)|}{2(x^2 + 1)}$$

$$= |x - 1| \cdot |x + 1| \cdot \frac{1}{2(x^2 + 1)}$$

$$< \delta \cdot \frac{5}{2} \cdot \frac{2}{5} = \delta < \varepsilon.$$

Therefore, $\lim_{x\to 1} \frac{1}{x^2+1} = \frac{1}{2}$.

Use definition prove that 9. **(10 marks)**

$$\lim_{x \to 1^+} \frac{x}{1 - x^2} = -\infty.$$

Proof. Let M < 0. Choose $\delta = \min \left\{ 1, -\frac{1}{3M} \right\}$. Then $0 < \delta \le 1$ and $0 < \delta \le -\frac{1}{3M}$.

It is equivalent to

$$-\frac{1}{3\delta} \le M.$$

Let $x \in \mathbb{R}$ such that $0 < x - 1 < \delta$. Then 0 < x - 1 < 1 or 1 < x < 2. So, 2 < x + 1 < 3. We obtain

$$\frac{1}{x-1} > \frac{1}{\delta} \quad \text{and} \quad \frac{1}{x+1} > \frac{1}{3}$$

It is clear that x - 1 > 0 and x + 1 > 0. Then

$$x > 1$$

$$x \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} > 1 \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} > \frac{1}{\delta} \cdot \frac{1}{3}$$

$$-x \cdot \frac{1}{x-1} \cdot \frac{1}{x+1} < -\frac{1}{\delta} \cdot \frac{1}{3}$$

$$\frac{x}{1-x^2} < -\frac{1}{3\delta} \le M.$$

Thus,
$$\lim_{x \to 1^+} \frac{x}{1 - x^2} = -\infty$$
.

10. (10 marks) Let f and g be functions on \mathbb{R} such that

$$g(x) > 1$$
 for all $x \in \mathbb{R}$

Let a be a limit point of $\mathbb R$ and $f(x) \neq 0$ for all $x \in \mathbb R$. Prove that if $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} \frac{g(x)}{|f(x)|} = +\infty.$$

Proof. Assume that f and g are functions on \mathbb{R} such that

$$g(x) > 1$$
 for all $x \in \mathbb{R}$

Let a be a limit point of \mathbb{R} and $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Suppose that $f(x) \to 0$ as $x \to a$.

Let M > 0. There is a $\delta > 0$ such that

$$0 < |x - a| < \delta$$
 implies $|f(x)| < \frac{1}{M}$.

Let $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$. We obtain |f(x)| > 0 and $\frac{1}{|f(x)|} > M$. From $g(x) \ge 1$, it follows that

$$\frac{g(x)}{|f(x)|} \ge \frac{1}{|f(x)|} > M.$$

Thus,
$$\frac{g(x)}{|f(x)|} \to +\infty$$
 as $x \to a$.