Final Examination Mathematical Analysis : SET A

Subject	Mathematical Analysis MAP2406	**Score**	100 marks
Time	8 a.m.-11 a.m. Friday 1 May 2020	**Semester**	2/2019
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D. Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University		

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1. **A(10 marks)** Let I be an open interval containing a, that, $f, g : I \to \mathbb{R}$. Prove that

   if f and g are **continuous** at a, then $f - g$ is **continuous** at a.

2. **A(10 marks)** Let $f(x) = \tan x \quad \text{where } x \in \left(0, \frac{\pi}{4}\right)$. Show that

   f is **uniformly continuous** on $\left(0, \frac{\pi}{4}\right)$.

   Hint: Use that fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.

3. **A(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

   $$\frac{x-1}{x} \leq \ln x \quad \text{for all} \quad x \geq 1.$$

4. **A(10 marks)** Use **L'Hospital's Rule** to estimate the limilt

   $$\lim_{x \to \infty} \left(1 + e^{-x}\right)^x.$$

5. **A(10 marks)** Let $a > 0$ and $f(x) = ax^2 + 1$ where $x \in [-1, 1]$. Suppose that

   $$U(f, P) - L(P, f) = 1 \quad \text{where} \quad P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}.$$

   What is a ?

6. **A(10 marks)** Let

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ 1 & \text{if } \quad 0 \le x \le 1 \end{cases}$$

Show that f is **integrable** on $[-1, 1]$

7. **A(10 marks)** Let $\displaystyle\int_{-1}^{0} f(t)\, dt = 2020$. Estimate the integral

$$\int_0^2 \frac{f\left(\frac{x-1}{x+1}\right)}{(x+1)^2}\, dx$$

8. **A(10 marks)** Use **Telescoping Series** to show that

$$\sum_{k=1}^{\infty} \frac{2^k}{(2^k+1)(2^{k+1}+1)} \qquad \text{converse and find its value.}$$

Hint: Use partial fraction

9. **A(10 marks)** Use the **Limit Comparision Test** to show that

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^p}\right) \qquad \text{converges} \quad \text{if } p > 1.$$

10. **A(10 marks)** Let $S_n = \displaystyle\sum_{k=1}^{n} a_k$ be a partial sum where

$$S_n = \frac{n}{n^2+1} \quad \text{for } n = 1, 2, 3, ...$$

Use **Dirichilet's Test** to prove that

$$\sum_{k=1}^{\infty} a_k \arctan\left(\frac{1}{k}\right) \qquad \text{converges.}$$

Final Examination Mathematical Analysis : SET B

Subject	Mathematical Analysis MAP2406	**Score**	100 marks
Time	11 a.m.-2 p.m. Friday 1 May 2020	**Semester**	2/2019
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D. Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University		

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1. **B(10 marks)** Let I be an open interval containing a, that, $f, g : I \to \mathbb{R}$. Prove that

   if f and g are **continuous** at a, then $2f + g$ is **continuous** at a.

2. **B(10 marks)** Let $f : I \to \mathbb{R}$ be uniformly continuous on I. Define

   $$g(x) = x + f(x) \quad \text{where } x \in I.$$

   Prove that g is **uniformly continuous** on I.

3. **B(10 marks)** Use the **Mean Value Theorem (MVT)** to prove that

   $$\sqrt{x+1} \leq \sqrt{x} + 1 \quad \text{for all} \quad x \geq 0.$$

4. **B(10 marks)** Let $f(x) = e^{-e^x}$ where $x \in \mathbb{R}$. Then f is 1-1 function.

   4.1 **B(5 marks)** Show that $f^{-1}(x) = \ln\left(\ln\left(\frac{1}{x}\right)\right)$ where $x \in (0,1)$

   4.2 **B(5 marks)** Use the **Inverse Function Theorem (IFT)** and 4.1 to find

   $$(f^{-1})'(x).$$

5. **B(10 marks)** Let $f(x) = x^4$ where $x \in [0,1]$. Find

   $$U(f,P) - L(P,f)$$

   in term of n when

   $$P = \left\{\frac{j}{n} : j = 0, 1, 2, ..., n\right\}.$$

6. **B(10 marks)** Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ 1 & \text{if } x \in (0, 2) \end{cases}$$

Show that f is **integrable** on $[0, 2]$

7. **B(10 marks)** Define

$$F(x) = \int_{\frac{1}{x}}^{x} \sin(e^t)\, dt$$

Find $F''(1)$.

8. **B(10 marks)** Show that

$$\sum_{k=3}^{\infty} \frac{k^4 + 4k^2 + 16}{k^6 - 64} \quad \text{converges and find its value.}$$

9. **B(10 marks)** Use the **Integral Test** to show that

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k + 1)^2} \quad \text{converges.}$$

10. **B(10 marks)** Prove that

$$\sum_{k=1}^{\infty} (-1)^k \arctan\left(\frac{1}{k}\right)$$

is conditionally convergent.

Hint: Use Alternating Series Test and Limit Comparision Test.

Solution Final Mathematical Analysis : SET C

Subject	Mathematical Analysis MAP2406 **Score** 100 marks
Time	2 p.m.-5 p.m. Friday 1 May 2020 **Semester** 2/2019
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D. Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University

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1. **C(10 marks)** Let I be an open interval containing a, that, $f, g : I \to \mathbb{R}$.
   Prove that

   if f and $f + g$ are **continuous** at a, then g is **continuous** at a.

2. **C(10 marks)** Let $f(x) = \dfrac{1}{1+|x|}$ where $x \in \mathbb{R}$. Show that

   f is **uniformly continuous** on $\mathbb{R}$.

3. **C(10 marks)** Let $p > 1$.
   Use the **Mean Value Theorem (MVT)** to prove that

   $$(x+1)^p \geq px + 1 \quad \text{for all} \quad x \geq 0.$$

4. **C(10 marks)** Let f and g be continuous on $[a, b]$ and differentiable on (a, b).
   Assume that $f(a) = f(b)$ and $g(a) = g(b)$.
   Use **Rolle's Theorem** to prove that there is a $c \in (a, b)$ such that

   $$f'(c) = g'(c).$$

5. **C(10 marks)** Define $f(x) = (x-1)^2 - 1$ where $x \in [0, 1]$. Find $I(f)$ if

   $$P = \left\{ \frac{j}{n} : j = 0, 1, ..., n \right\}.$$

6. **C(10 marks)** Let

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$$

Show that f is integrable on $[0, 2]$

7. **C(10 marks)** Let $f(x) = \int_0^{x^2} \sec^2(t^2)dt$. Use **integration by part** to show that

$$2\int_0^1 \sec^2(x^2)\,dx - 4\int_0^1 xf(x)\,dx = \tan 1.$$

8. **C(10 marks)** Find a **partial sum** S_n of

$$\sum_{k=1}^{\infty} \frac{2k-1}{2^k}$$

and show that it **converges**.

Hint: The idea is similar to geometric seires proof.

9. **C(10 marks)** Use the **Ratio Test** to find all of $x \in \mathbb{R}$ such that Bessel function of first order $J_1(x)$ **converges** where

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k+1}}.$$

10. **C(10 marks)** Assume that $\sum_{k=1}^{\infty} a_k$ coverges absolutely.

Use **Cauchy Criterion** to prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{k} \quad \text{coverges absolutely.}$$

Solution Final Mathematical Analysis : SET D

Subject	Mathematical Analysis MAP2406	**Score**	100 marks
Time	9 a.m.-12 a.m. Monday 4 May 2020	**Semester**	2/2019
Teacher	Assistant Professor Thanatyod Jampawai, Ph.D. Division of Mathematics, Faculty of Education, Suan Sunandha Rajabhat University		

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1. **D(10 marks)** Let $a \in \mathbb{R}$ and $f(x) = \dfrac{1}{x^2+1}$ where $x \in \mathbb{R}$. Prove that

$$f \text{ is } \textbf{continuous} \text{ at } a.$$

2. **D(10 marks)** Let I be an open interval, that, $f, g : I \to \mathbb{R}$.
Prove that if f and g are **uniformly continuous** on I, then

$$f + g \text{ is } \textbf{uniformly continuous} \text{ on } I.$$

3. **D(10 marks)** Let $0 < p < 1$.
Use the **Mean Value Theorem (MVT)** to prove that

$$(x+1)^p \le px + 1 \quad \text{for all} \quad x \ge 0.$$

4. **D(10 marks)** Let $f(x) = \dfrac{e^x - e^{-x}}{2}$ where $x \in \mathbb{R}$. Then f is 1-1 function.

   4.1 **D(5 marks)** Show that $f^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ where $x \in \mathbb{R}$.
   Hint: Write out in term of the perfect square.

   4.2 **D(5 marks)** Use the **Inverse Function Theorem (IFT)** and 4.1 to find

$$(f^{-1})'(x).$$

5. **D(10 marks)** Let $n \in \mathbb{N}$ and define $f : [0, n] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 4 & \text{if} \quad 1 \le x < 2 \\ 9 & \text{if} \quad 2 \le x < 3 \\ \vdots & \qquad \vdots \\ n^2 & \text{if} \quad (n-1) \le x \le n \end{cases}$$

If $\displaystyle\int_0^n f(x)\,dx = 385$, what is n.

6. **D(10 marks)** Let

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 0 \\ 0 & \text{if } \quad 0 < x \le 1 \end{cases}$$

Show that f is integrable on $[-1, 1]$

7. **D(10 marks)** Let $f(x) = e^{\sin(\pi x)}$. Estimate the integral

$$\int_0^1 x f''(x)\, dx.$$

8. **D(10 marks)** Show that the below seires is **converges and find it value**:

$$\sum_{k=0}^{\infty} \left[\frac{1}{3^{1+k} \cdot 2^{1-k}} + \frac{1}{k^2 + 4k + 3} \right]$$

9. **D(10 marks)** Use the **Root Test** to find all of $x \in \mathbb{R}$ such that

$$\sum_{k=1}^{\infty} \left(\frac{(kx+1)^2}{k^2+1} \right)^k \qquad \text{converges.}$$

10. **D(10 marks)** Use **Dirichilet's Test** to prove that

$$S(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

converges for all $x \in \mathbb{R}$.