History of Mathematics

Chapter 3

The period from 1000 B.C to 300 B.C.

Topics

- The Greeks
- Origins of Greek Mathematics
- From Pythagoras to Plato
- Influence of Plato and Aristotle
- The Orient

1. The Greeks

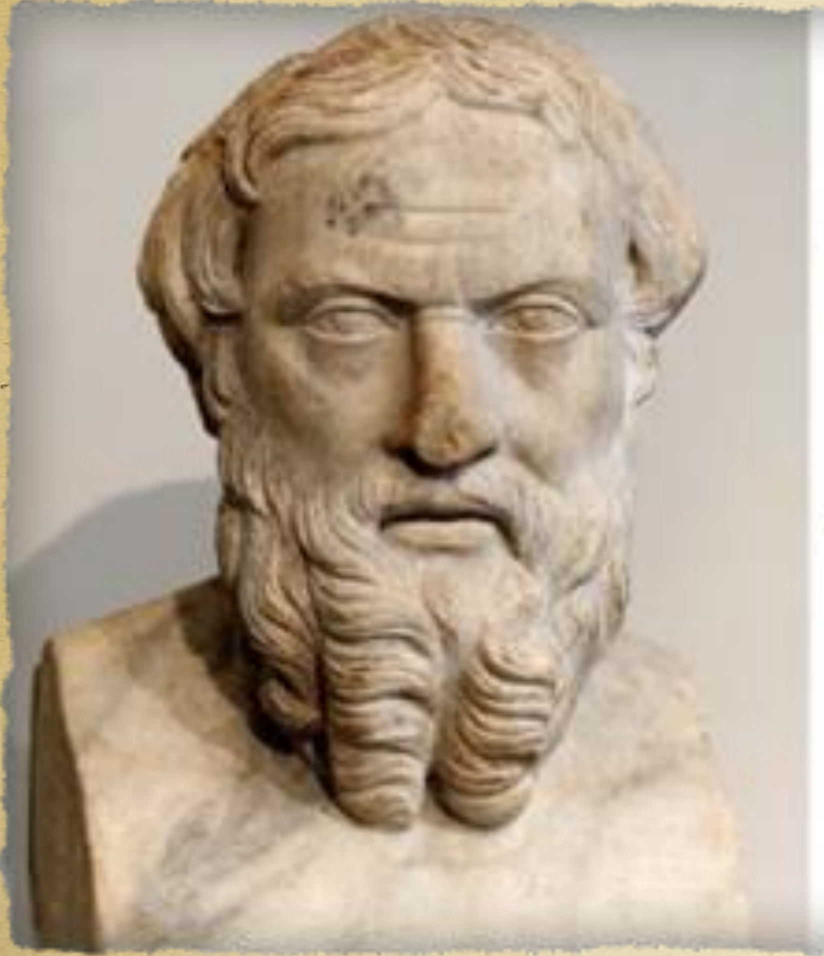
Birth of Greek Arithmetic

Commercial Arithmetic

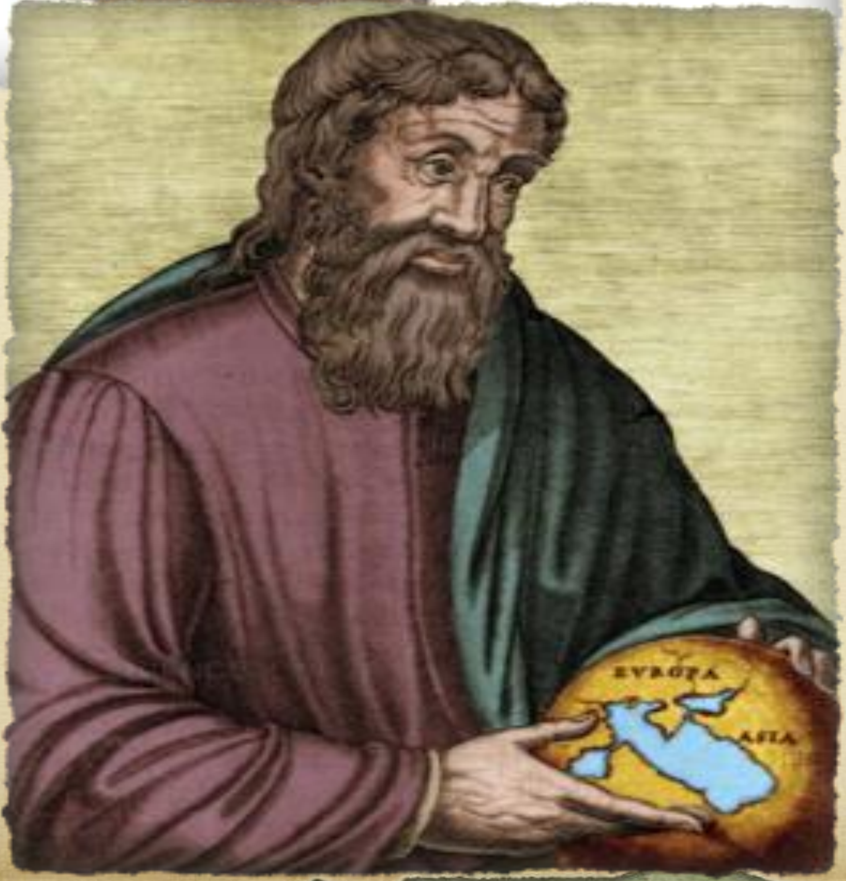
The merchants of Phoenician coast

Babylon
Egypt
Asia Minor

Herodotus



Strabo



Miletus

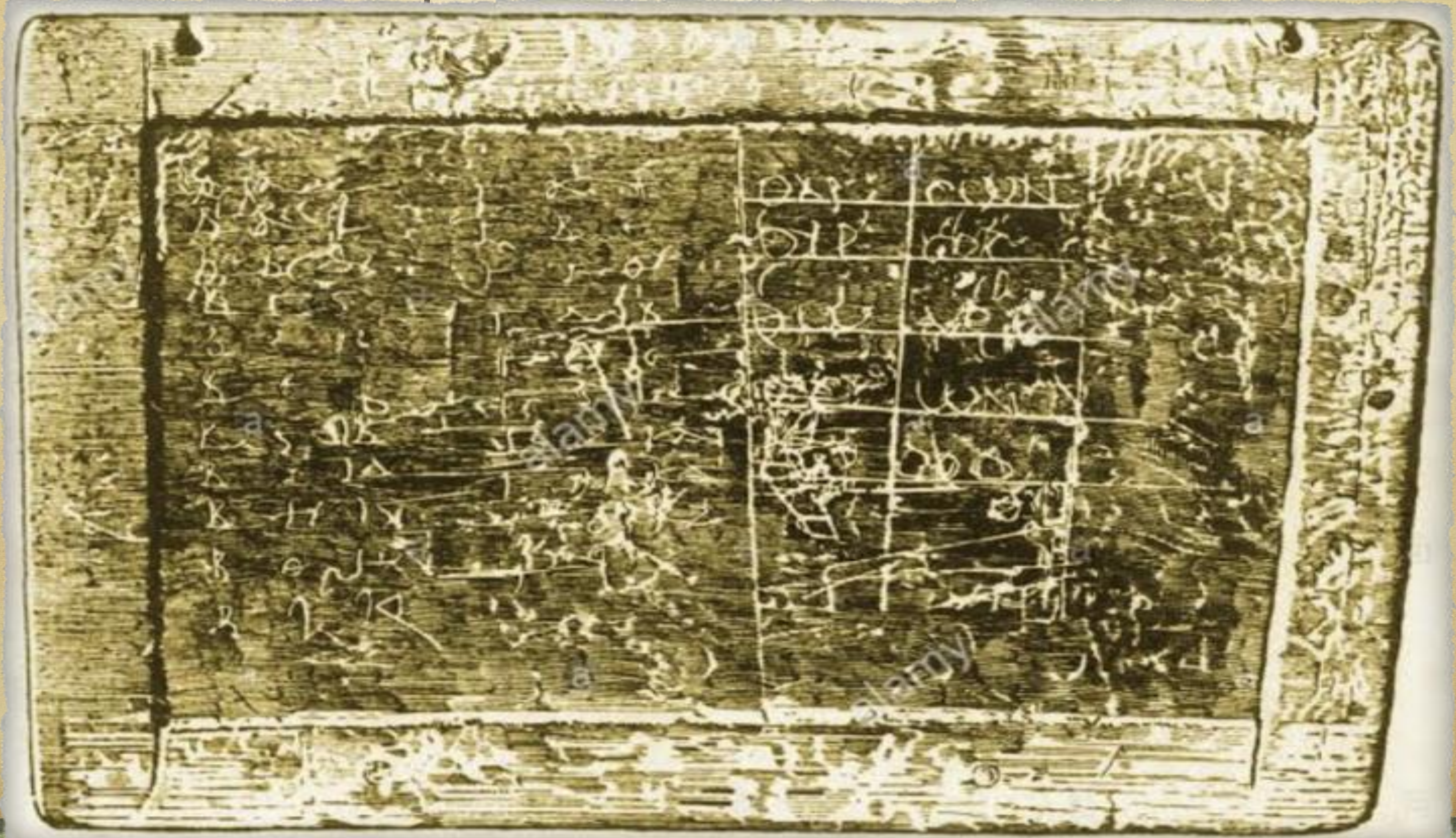
The Greatest
Commercial Town

Miletus

ANCIENT COINS

Coins found in Asia Minor. They are among the earliest known, dating from about 550 B.C. From Breasted's *Ancient Times*

Greek multiplication table on a wax tablet



Centers of Mathematics Activity

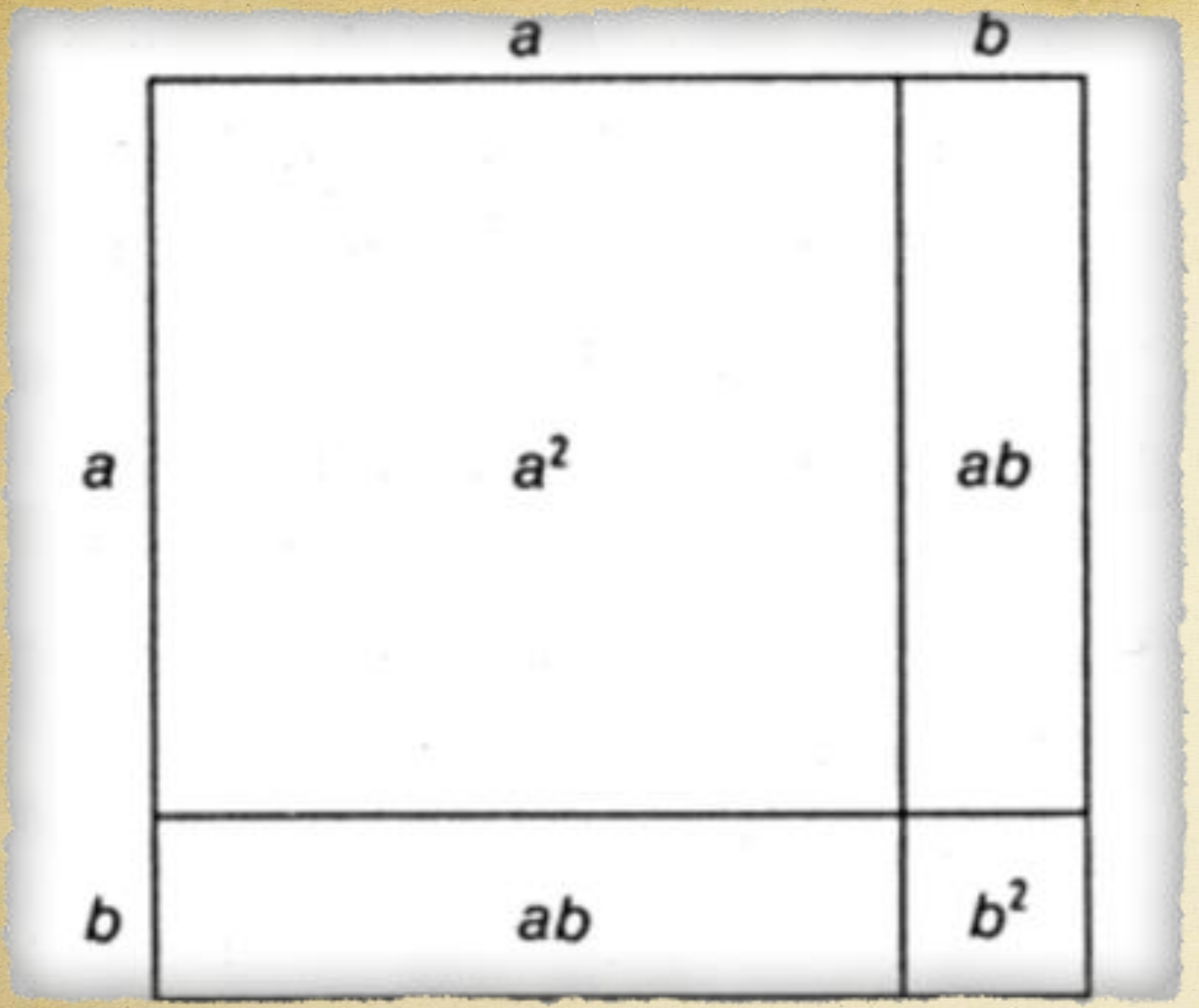
Abdera, 16.
Alexandria, 41.
Amisus, 31.
Antinoopolis, 43.
Apameia, 35.
Aquitania, 2.
Athens, 13.
Byzantium, 28.
Cadiz, 1.
Chalcedon, 29.
Chalcis, 37.
Chios, 17.
Clazomenæ, 24.
Cnidus, 22.
Constantinople, 28.
Crete, 20.
Crotona, 9.
Cyprus, 34.
Cyrene, 11.
Cyzicus, 27.
Elea, 8.
Elis, 12.
Gades, 1.
Gerasa, 39.
Jerusalem, 40.
Laodicea, 32.
Larissa, 36.
Medma, 7.
Mendes, 42.
Miletus, 23.
Naples and Pompeii, 6.
Nicæa and Bithynia, 30.
Paros, 19.
Perga, 33.
Pergamum, 26.
Ptolemais, 44.
Rhodes, 21.
Rome, 3.
Samos, 18.
Sicily, 4.
Smyrna, 25.
Stageira, 14.
Syene, 45.
Syracuse and Messina, 5.
Tarentum, 10.
Thasos, 15.
Tyre, 38.

GREEK GEOMETRIC FORMS JUST PRECEDING THE TIME OF THALES

From a specimen of the 8th century B.C. in the Metropolitan Museum, New York

Early Greek Appreciation of Geometric Forms

Greek Algebra



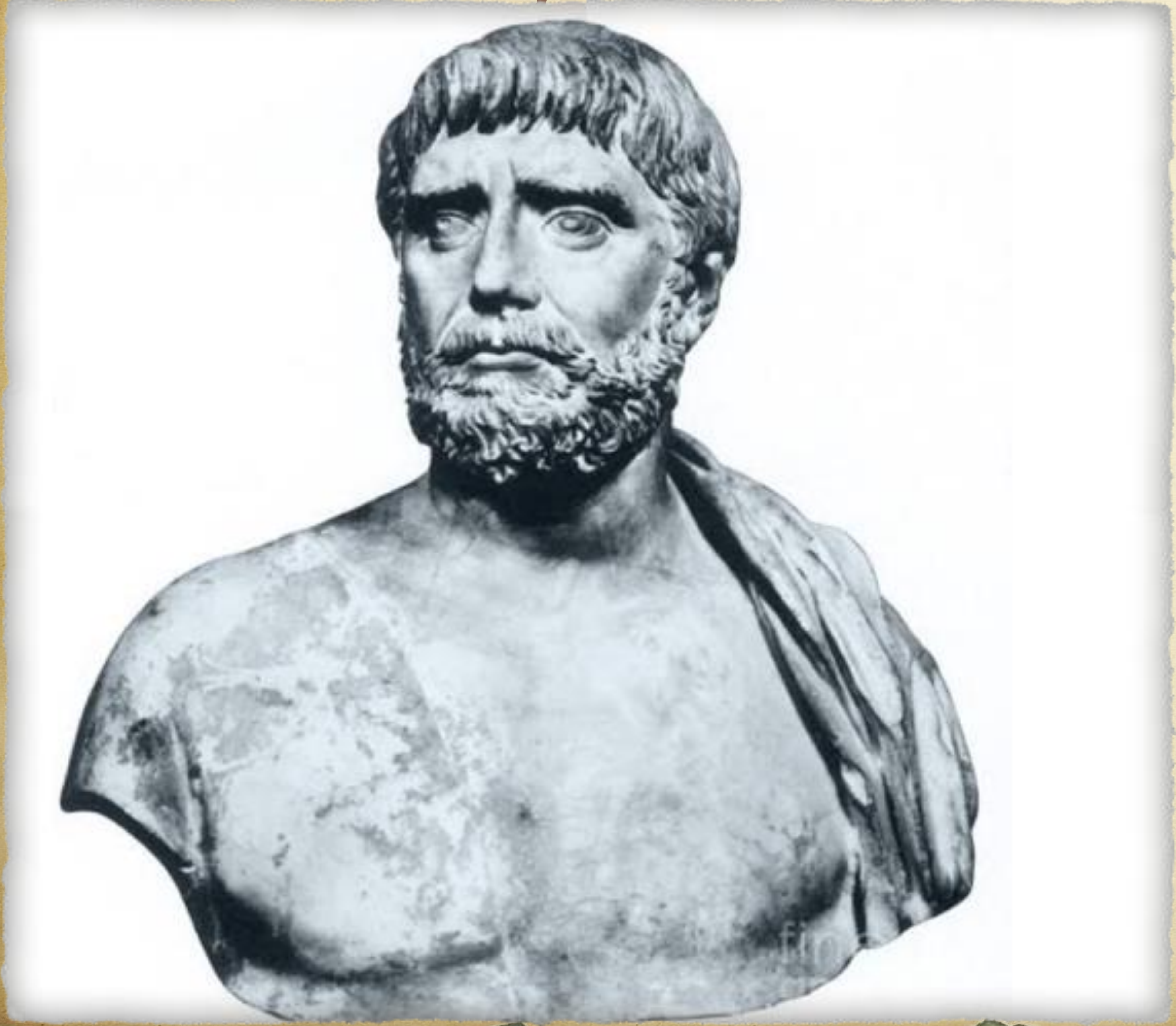
$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$(a+b)(a-b) = a^2 - b^2,$$

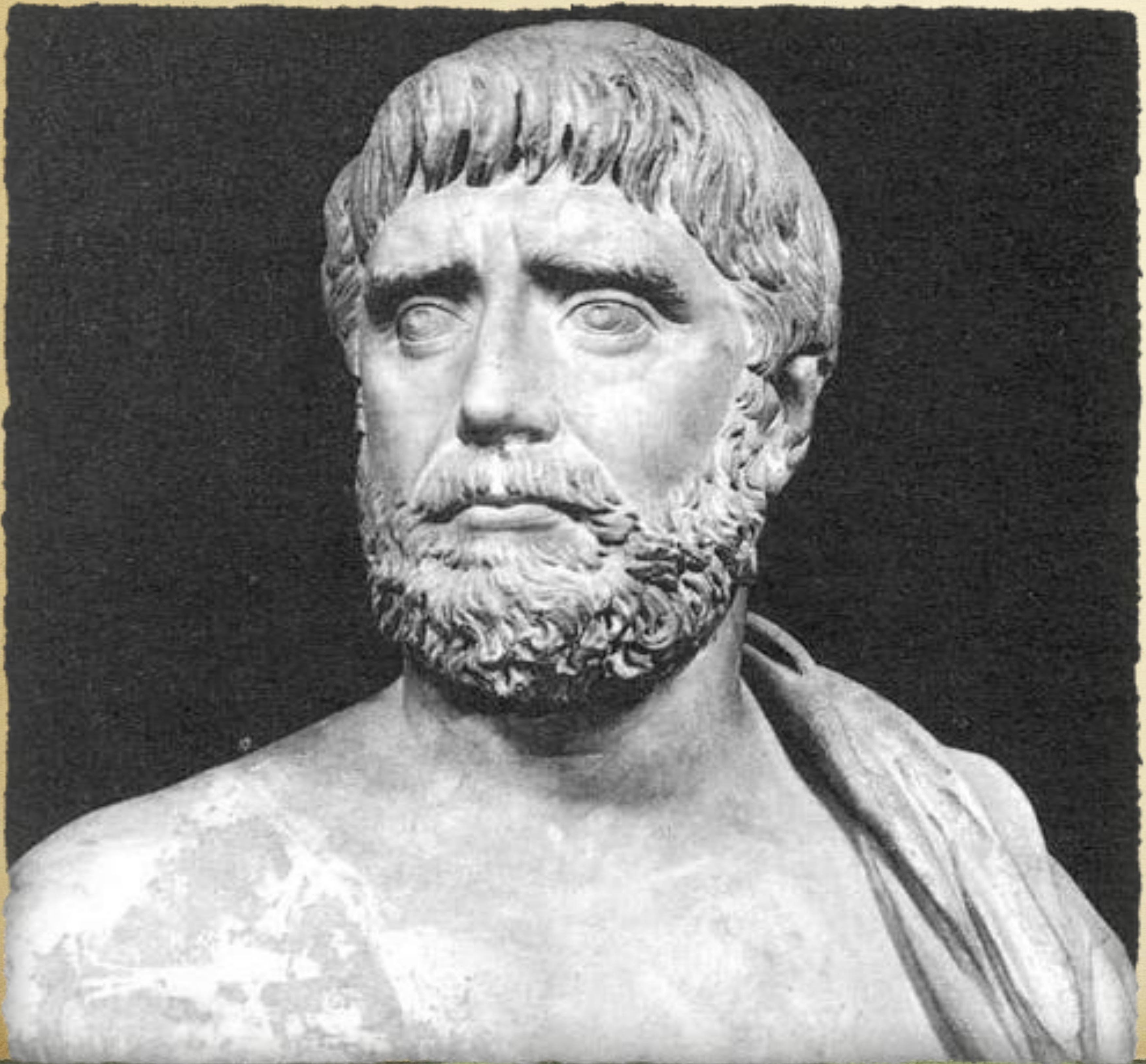
$$a(x+y+z) = ax + ay + az,$$

and $$(a-b)^2 = a^2 - 2ab + b^2,$$

2. Origins of Greek Mathematics



Thales



The first of the Greek to take any scientific interest in mathematics in general.

Astronomy
Geometry
Theory of number

Stories concerning Thales

—evidently that of the owner.[1] Considering the general recognition of the abilities of Thales, even during his lifetime, it is not impossible that a vase made in Cyprus may have been intended for him, but there is no further evidence that this was the case.

CYPRIOTE VASE WITH THE NAME OF THALES

The name *TA + LE + SE*, when viewed from above, has the appearance shown in the text. The vase is about contemporary with Thales of Miletus. From the Metropolitan Museum of Art, New York

Stories concerning Thales. Thales was a merchant in his younger days, a statesman in his middle life, and a mathematician, astronomer, and philosopher in his later years. In his mercantile ventures he seems to have been unusually successful, even in dealing with the shrewdest of the Greek trading races. Aristotle (*c.* 340 B.C.) tells us how he secured control of all the oil presses in Miletus and Chios in a year when olives promised to be plentiful, subletting them at his own rental when the season came. Plutarch (1st century) also testifies to his ingenuity in the following anecdote:[2]

Solon went, they say, to Thales at Miletus, and wondered that Thales took no care to get him a wife and children. To this Thales made no answer for the present; but a few days after procured a stranger to pretend that he had left Athens ten days earlier and

Arithmetic of Thales

Arithmetic of Thales. Of the nature of the arithmetic that Thales brought back from Egypt we have little direct knowledge. Iamblichus of Chalcis (*c.* 325 A.D.) tells us that he defined number as a system of units, and adds that this definition and that of unity came from Egypt. This is not much, but it is enough to show that Thales was interested in something besides the merely practical. It is probable that he knew many

other number relations, for the Ahmes papyrus contains some work in progressions, and such knowledge would hardly escape so careful an observer as Thales. It is, however, in his work in founding deductive geometry and in his capacity as a teacher of Pythagoras rather than as a discoverer of facts that Thales commands our attention.

Geometry of Thales

Geometry of Thales. In geometry he is credited with a few of the simplest propositions relating to plane figures. The list, according to the most reliable ancient writers, is as follows:

1. Any circle is bisected by its diameter.
2. The angles at the base of an isosceles triangle are equal.
3. When two lines intersect, the vertical angles are equal.[1]
4. An angle in a semicircle is a right angle.[2]

5. The sides of similar triangles are proportional.[1]
6. Two triangles are congruent if they have two angles and a side respectively equal (Euclid, I, 26).[2]

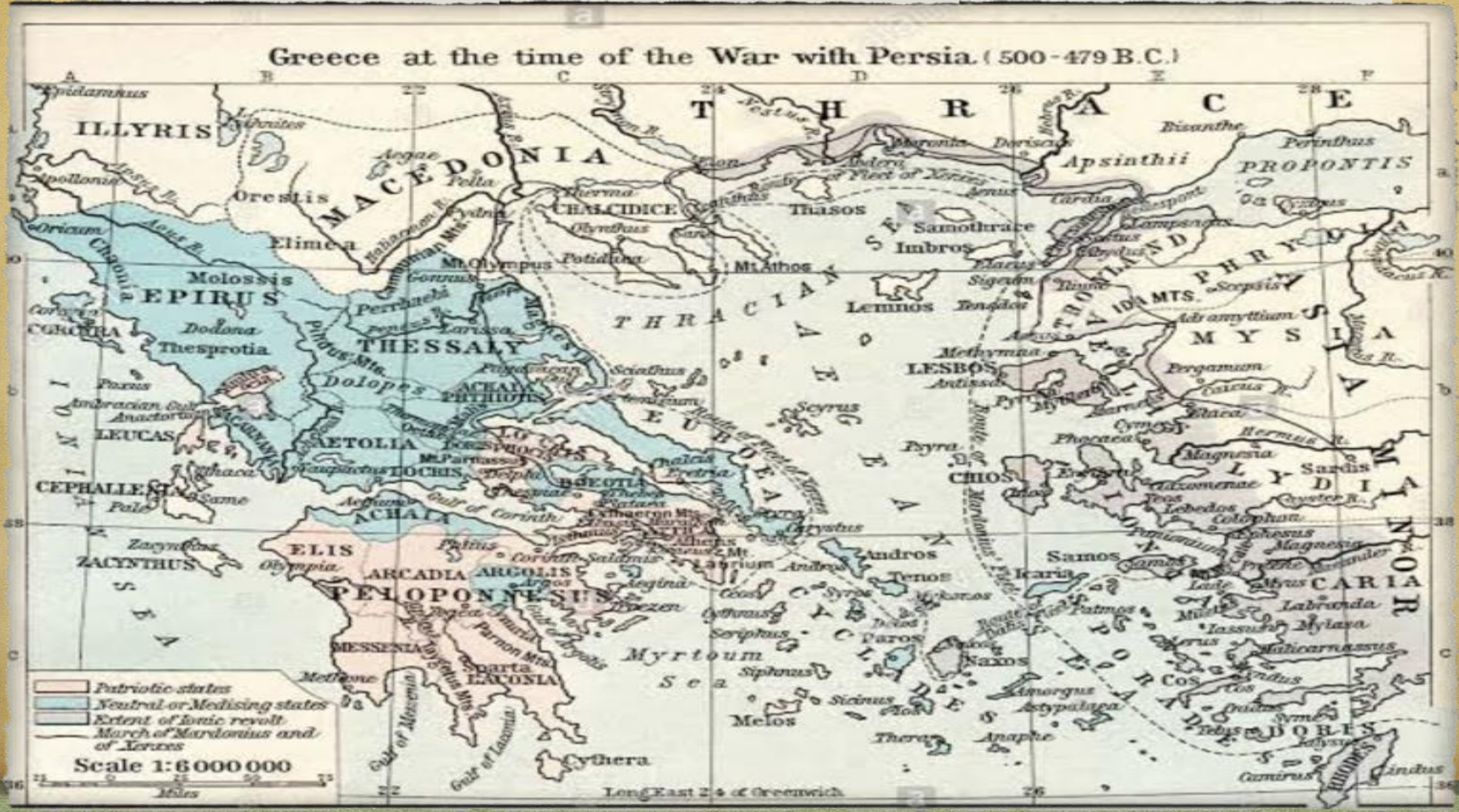
3. From Pythagoras to Plato

Pythagoras of Samos

FIGURE OF PYTHAGORAS

A coin of Samos of the reign of Trajan (98–117), and therefore much later than Pythagoras. It shows the honor in which he was held and the claim of Samos as his native country

Samos


Greece at the time of the War with Persia. (500-479 B.C.)
THRACE
MACEDONIA
ILLYRIS
EPIRUS
THESSALY
AETOLIA
ACHAIA
ELIS
ARCADIA
ARGOLIS
PELOPONNESUS
MESSENIA
LACONIA
EUBOEA
BOEOTIA
THRACIAN SEA
AEGEAN SEA
Myrtoum Sea
CYCLADES
SPORADES
Thasos
Samothrace
Imbros
Lemnos
LESBOS
CHIOS
Samos
Icaria
Naxos
Paros
Andros
Tenos
Melos
Thera
Cythera
Cos
RHODES
MYSIA
LYDIA
CARIA
PHRYGIA
ASIA MINOR
PROPONTIS
Sardis
Patriotic states
Neutral or Medising states
Extent of Ionic revolt
March of Mardonius and of Xerxes
Scale 1:6 000 000

School of Pythagoras

School of Crotona

Philosophy of Pythagoras

Philosophy of Pythagoras. Pythagoras based his philosophy upon the postulate that number is the cause of the various qualities of matter. This led him to exalt arithmetic, as distinguished from logistic, out of all proportion to its real importance. It also led him to dwell upon the mystic properties of numbers and to consider arithmetic as one of the four degrees of wisdom,—arithmetic, music, geometry, and spherics (astronomy), these forming the quadrivium of the Middle Ages. Aristotle (384–322 B.C.) tells us that Pythagoras related the virtues to numbers, and Plutarch says that he believed that earth was produced from the regular hexahedron, fire from the pyramid, air from the octahedron, water from the icosahedron, and the heavenly sphere from the dodecahedron, in all of which the physical elements are related both to number and to form.

Geometry of Pythagoras

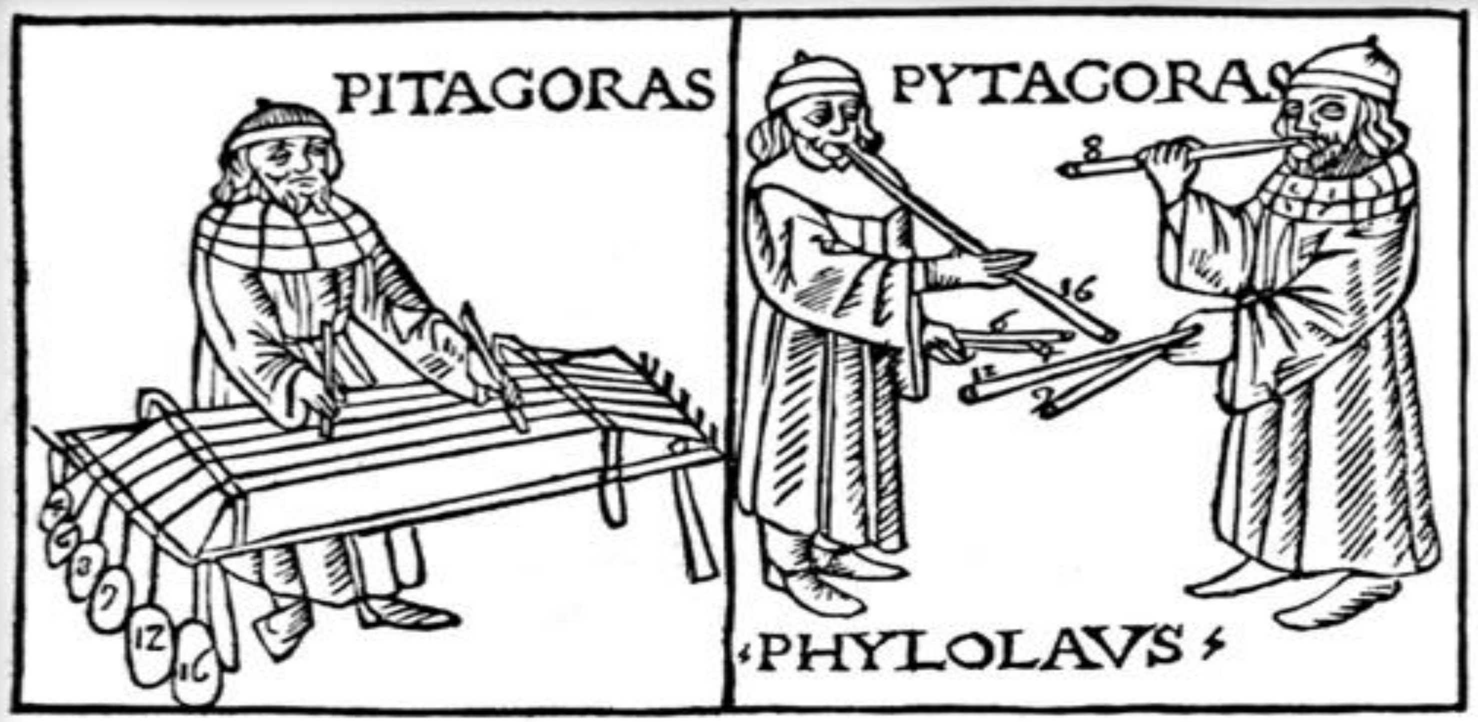
Geometry of Pythagoras. In the field of geometry Eudemus (*c.* 335 B.C.) informs us that Pythagoras "investigated his theorems from the immaterial and intellectual point of view," and that "he discovered the theory of irrational quantities and the construction of the mundane figures."[1] Favori′nus, a philosopher living in southern France *c.* 125, asserts that he employed definitions in his work in mathematics, this being the first trace that we have of such use.[2] In particular, he defined a point as "unity having position."[3] He or his school knew that the plane space about a point may be filled by six equilateral triangles, four squares, or three regular hexagons,—a fact which had doubtless been inferred as a result of the observation of mosaic pavements long before this time, but which no doubt he was able to prove. It is probable that Pythagoras proved the proposition relating to the sum of the angles of a triangle, that he constructed a polygon equivalent to one given polygon and similar to another, and that he could construct the five regular polyhedrons; and he may possibly have proved the theorem relating to the square on the hypotenuse. It seems likely that he taught that the earth is a sphere in space; at any rate, this theory was accepted by various later philosophers.[4]

Pythagoras on Music

Pythagoras on Music. Pythagoras is said to have discovered that the fifth and the octave of a note can be produced on the same string by stopping at $\frac{2}{3}$ and $\frac{1}{2}$ of its length, respectively, and it is thought that this harmony gave rise to the name of "harmonic proportion," since

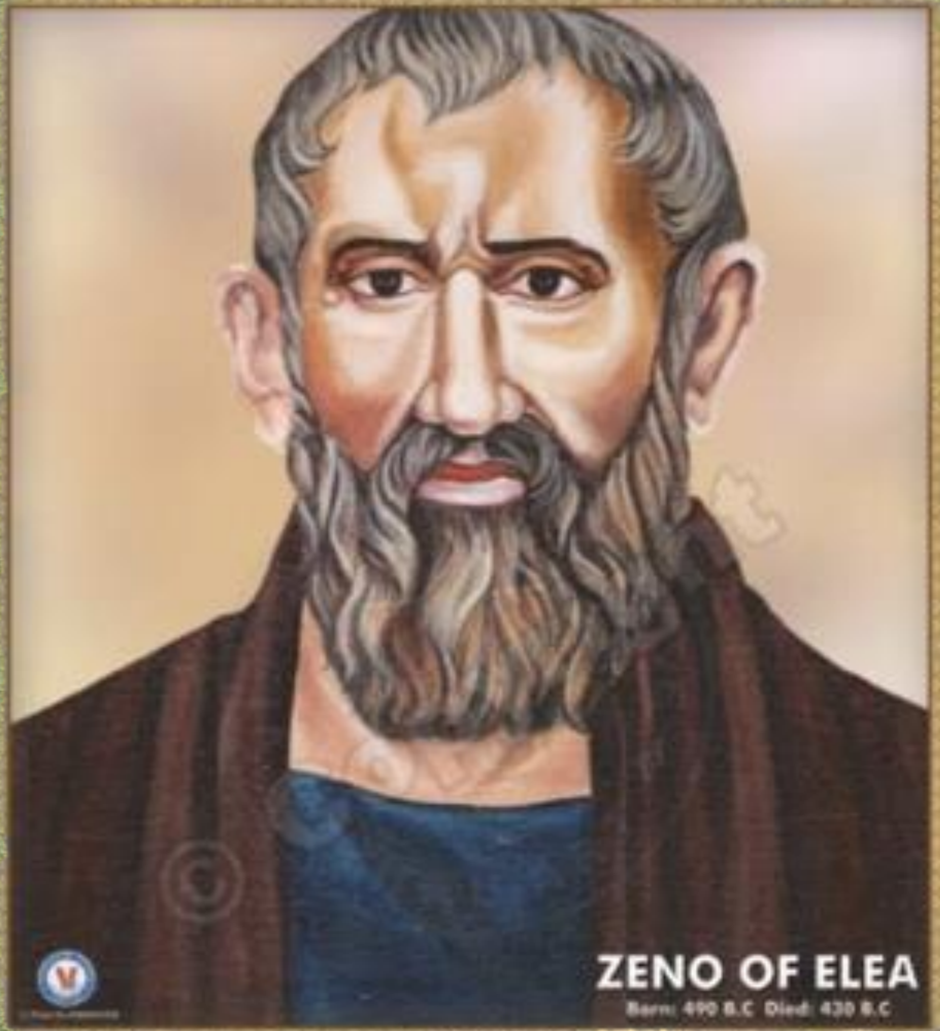
$$1 : \frac{1}{2} = 1 - \frac{2}{3} : \frac{2}{3} - \frac{1}{2}.$$

Although he seems to have derived some knowledge of music from Egypt,[5] he is generally called the inventor of musical science, or the harmonic canon (a mere tradition), but we



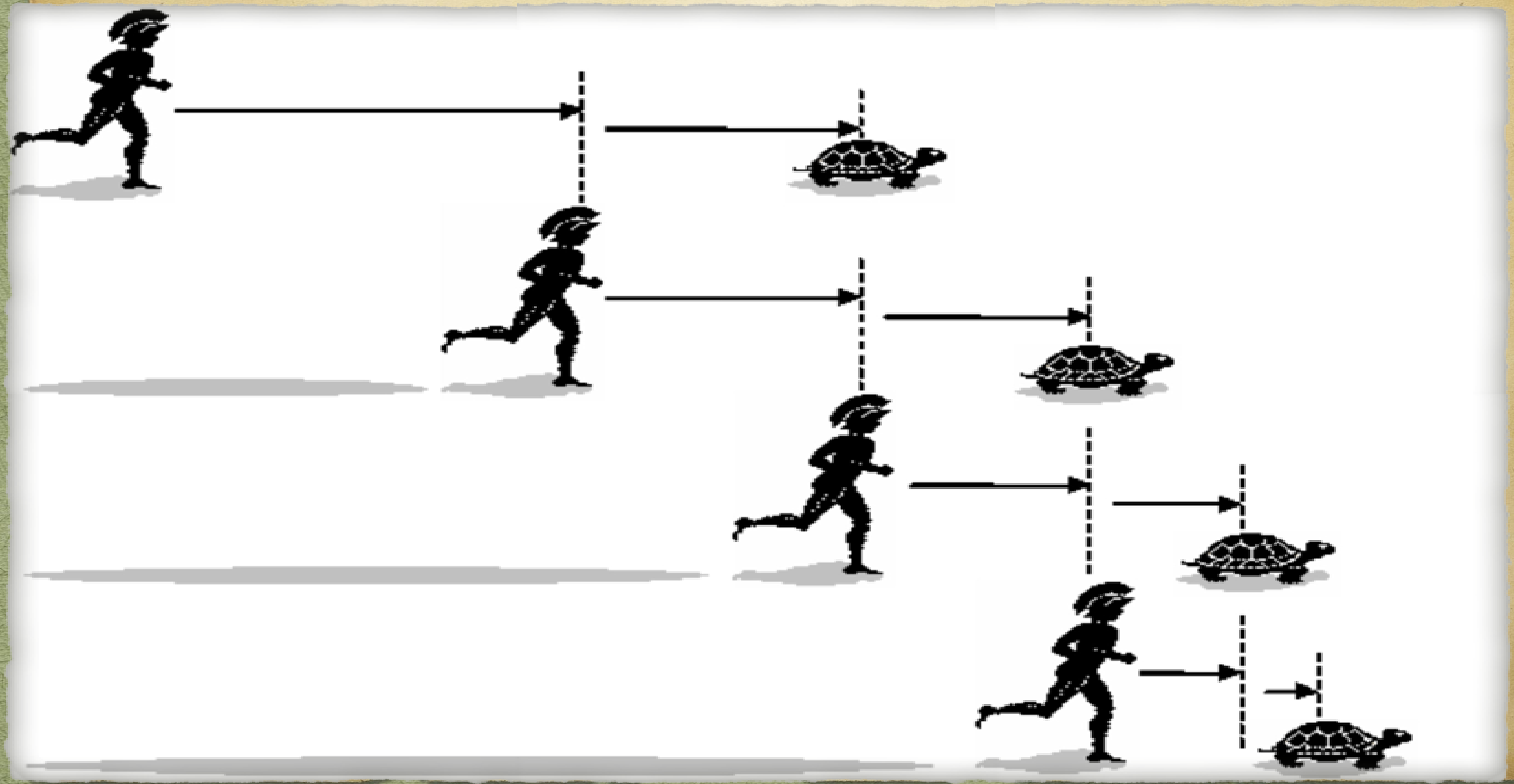
PYTHAGORAS THE MUSICIAN

From F. Gafurius, *Theorica Musice*, Milan, 1492. One of the first crude attempts to portray Pythagoras by means of a woodcut, and the first to portray him as a musician. In the same work he is also shown as a bell ringer

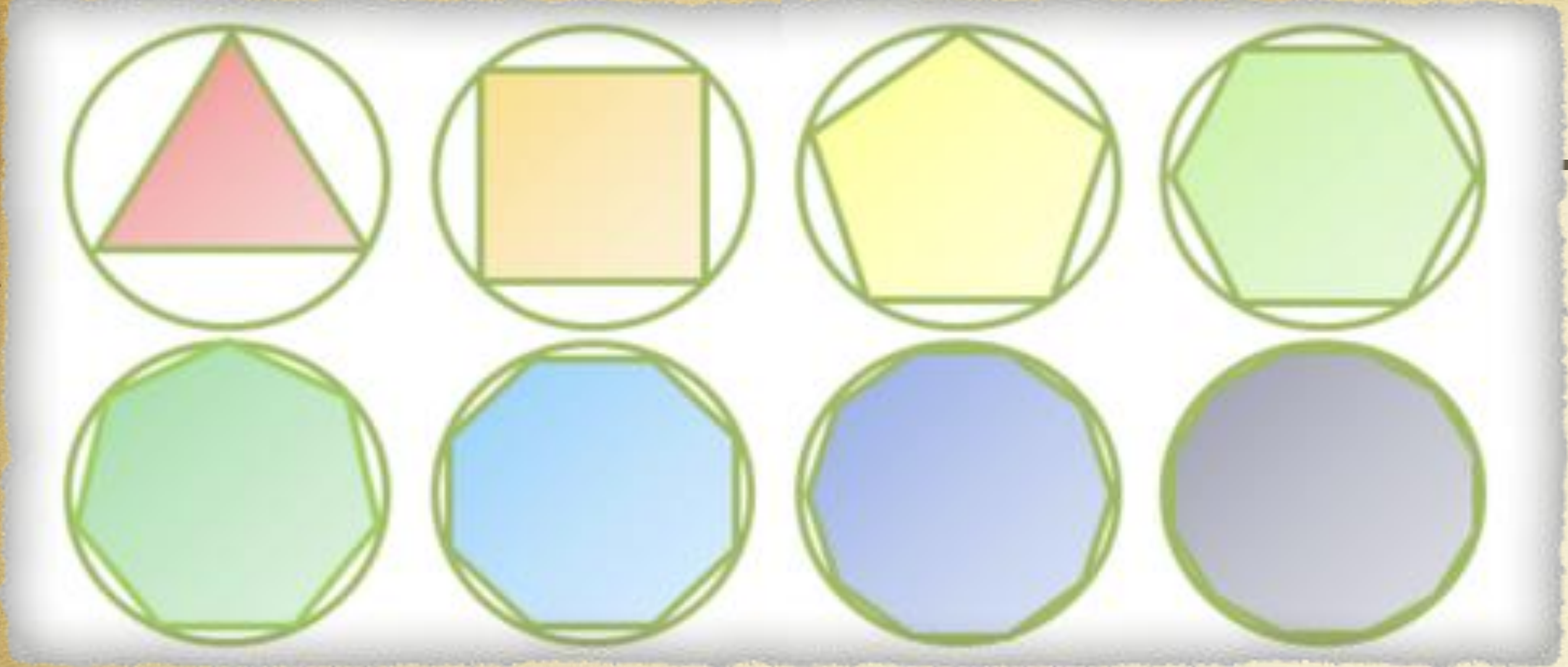


Zeno of Elea

Achilles and Tortoise



The Method of Exhaustion



The Method of Exhaustion. Iamblichus (*c.* 325) mentions Bry'son,[1] or Bryso, as one of the youths whom Pythagoras instructed in his old age. If this is true, Bryson must have been born about 520 B.C., but it is commonly believed that he flourished about 450 B.C. He was formerly thought to have contributed to what is known as the method of exhaustion, a crude approach to the integral calculus whereby the area between a curvilinear figure (say a circle) and a rectilinear figure (say an inscribed regular polygon) could be approximately exhausted by increasing the number of sides of the latter. There is, however, no reliable ancient authority for connecting his name with the theory.[2] The method was effectively used by later writers, notably by Eudoxus and Archimedes, and was extended to include the mensuration of solids.

Eudemus (335 B.C.)

Eudemus (*c.* 335 B.C.), speaking of his work in geometry, tells us that he was one of those who "enriched the science with original theorems and gave it a sound arrangement," and from another statement we infer that he knew and doubtless proved the following propositions:

1. If a perpendicular is drawn to the hypotenuse from the vertex of the right angle of a right-angled triangle, each side is the mean proportional between the hypotenuse and its adjacent segment.
2. The perpendicular is the mean proportional between the segments of the hypotenuse.
3. If the perpendicular from the vertex of a triangle is the mean proportional between the segments of the opposite side, the angle at the vertex is a right angle.
4. If two chords intersect, the rectangle of the segments of one is equivalent to the rectangle of the segments of the other.
5. Angles in the same segment of a circle are equal.
6. If two planes are perpendicular to a third plane, their line of intersection is perpendicular to that plane and also to their lines of intersection with that plane.[1]

4. Influence of Plato and Aristotle

Mysticism of Numbers

Mysticism of Numbers. One thing that particularly interested Plato was the mysticism of numbers. In his *Republic* (Book VIII) he speaks in an obscure fashion of a certain mystic number, but does not make it clear what this number is. He calls it "the lord of better and worse births," and subsequent writers have often tried to find exactly what he meant. One theory is that 60^4, or 12,960,000, is the Platonic number. This number played an important part in the mysticism of the Hindus and the Babylonians, and it is possible that Pythagoras found it on the banks of the Euphrates, if he really studied there, and that he took it with him to Crotona, passing it on to his disciples, who, in turn, told it to Plato and his followers.

Although Plato esteemed the science of numbers highly,[2] he gives us no information concerning the way it was taught in his school or what it included. We are about as ignorant of the subject as presented by him, and of the ground it covered, as we are of the ancient logistic.[3]

Plato on Geometry

Plato on Geometry. More than any of his predecessors Plato appreciated the scientific possibilities of geometry, of which more will be said in Volume II of this work. By his teaching he laid the foundations of the science, insisting upon accurate definitions, clear assumptions, and logical proof. His opposition to the materialists, who saw in geometry only what was immediately useful to the artisan and the mechanic, is made clear by Plutarch (1st century) in his *Life of Marcellus*. Speaking of the use of mechanical appliances in geometry, Plutarch remarks upon "Plato's indignation at it and his invectives against it as the mere corruption and annihilation of the one good of geometry, which was thus shamefully turning its back upon the unembodied objects of pure intelligence." That Plato should hold the view here indicated is not a cause for surprise. The world's thinkers have always held it. No man ever created a mathematical theory for practical purposes alone. The applications of mathematics have generally been an afterthought.[1]

Eudox'us of Cnidus

Eudox'us of Cnidus,[4] at one time a pupil of Plato, achieved eminence in astronomy, geometry, medicine, and law.[5] It is said that he introduced the study of spherics (mathematical astronomy) into Greece and made known the length of the year as he had found it given in Egypt. He was the first of the Greeks, so far as is known, to give a description of the constellations. Strabo asserts that the observatory of Eudoxus still existed at Cnidus in his time, that is, about the beginning of the Christian era. Seneca says that he brought from Egypt to Greece the theory of the motions of the planets; Aristotle records that he made separate spheres for the stars, sun, moon, and planets; and Archimedes says that he found the diameter of the sun to be nine times that of the earth and showed that a pyramid is one third of a prism of the same base and the same altitude, and similarly for a cone and cylinder. For the mensuration of the cone and cylinder he probably developed the method of exhaustion as a rigorous theory.[6] Vitruvius gives him credit for a new form of sundial called the spider's web,[7] which may, however, have been an astrolabe. Because of a note, possibly due to Proclus, he is often credited with having written a work on proportion which finally became Book V of Euclid, but for this statement there is no definite historical

Aristotle

Ar′istotle[2] studied under Plato at Athens, and his diligence and brilliancy led the latter to call him the "intellect of the school."[3] He became one of the instructors of Alexander the Great, and later returned to Athens and founded the Peripatetic School of philosophy, probably so called from the place where he taught.[4] He was a voluminous writer, but although many of his works are extant the major part are lost. His interest in the mathematical sciences lay chiefly in their applications to physics. He speaks of mathematics as standing halfway between physics and metaphysics. He wrote two works of a mathematical nature, one on indivisible lines and the other on mechanical problems. Both have been edited and printed. We know that, contrary to the doctrines of the Pythagoreans, he advocated the separation of arithmetic and geometry. In his systematizing of logic he contributed indirectly to the great work of Euclid. To him, too, we owe the first known definition of continuity: "A thing is continuous when of any two successive parts the limits at which they touch are one and the same and are, as the word implies, held together."[5] Aristotle was also interested in the historical development of science, and this seems to have influenced the work of his disciples in

5. The Orient

China

China. It is an interesting fact that Egypt developed a worthy type of mathematics before 1000 B.C. and then stagnated, that Babylonia did the same, and that China followed a similar course. Was it that the world's vigor was concentrated in Greece? Had the older civilizations burned out? Or was there some subtle influence that subjected the original seats of mathematical thought to canonical expression instead of progressive action? Whatever the answer, between 1000 and 300 B.C. China produced no great classic in mathematics, unless possibly the *Nine Sections*[2] already mentioned, or the *Wu-ts'ao Suan-king* to be mentioned later, belongs to this period. It was rather in the impetus given to commercial calculation through the introduction of coins in the 7th century B.C., at about the same time as they appeared in Asia Minor, that China made her most noteworthy contribution to the progress of arithmetic. Knife money and spade money appeared *c.* 670 B.C., the coins representing such common articles of value as knives and spades. Circular coins were issued later and became the standard forms in the 3d century B.C. As to the methods used in calculating at this time, we are ignorant, but some mechanical means were probably employed in China as well as in other parts of the ancient world. About 542 B.C. the Chinese are known to have used in their calculations bamboo rods, in size and appearance somewhat like a new lead

India

India. As already stated, we have no authentic records of India before the Mohammedan invasion (7th century), almost our only sources of information being the Vedic literature, the Buddhist sacred books, the heroic poems, such inscriptions as remain on monuments, and the metal land grants. Of these, the later Vedic literature, the heroic poems, and the Buddhist writings are all that give us any knowledge of the mathematics of the period from 1000 to 300 B.C. The Vedic writings probably extend down to about 800 B.C., although the *Vedāngas* ("Limbs for supporting the Veda") were written several centuries later. The dates of the *Śulvasūtra* period are unknown. Taking the opinions of various scholars and forming a rough estimate, we may put the ritualistic rules of the *Śulvasūtras* in the five centuries just preceding our era. The rules which have any mathematical interest relate indirectly to the proportions of altars in the temples. They include a statement about Pythagorean numbers, that is, numbers satisfying the relation $x^2 + y^2 = z^2$, and imply a statement of the Pythagorean Theorem itself. There is no reason for believing, however, that the Hindus had the slightest idea of the nature of a geometric proof. There is also evidence of a knowledge of irrationals and

of an understanding of the uses of the gnomon.[1] The *Śulvasūtras* also state that the diagonal of a unit square is equal to

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34},$$

or 1.4142156. The area of the circle is asserted[2] to be

$$\left(\frac{1}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29 \cdot 6} + \frac{1}{8 \cdot 29 \cdot 6 \cdot 8}\right)^2 d^2.$$

Mesopotamia

By 1000 B.C. they had developed a system of alphabetic writing, and their bills of exchange were known in Mesopotamia, Persia, and India, as those of Babylon had been known before them. All through this early period the records of taxes show that this form of applied arithmetic was ever present.

In the 8th century B.C. the Assyrians subdued Mesopotamia and much of the territory to the west and became the dominating power in Western Asia. They maintained the first great army equipped with weapons of iron and by this means held a large territory in subjection. Militarism, however, eventually proved a weakness, and they in turn succumbed to

HEBREWS PAYING TRIBUTE TO SHALMANESER III, KING OF ASSYRIA

This was about 850 B.C. The original is now in the British Museum. From Breasted's *Ancient Times*

Contributions of Babylonia and Assyria

Contributions of Babylonia and Assyria. In the midst of all these changes two steps in the history of mathematics deserve special mention: (1) the Arameans brought the arithmetic of commerce to a higher standard, and (2) the Babylonians and Chaldeans extended the earlier work in astronomy. The science of astrology had by this time developed as a potent force in civilization, and astronomy had become recognized as the science par excellence. Ptolemy the astronomer (*c.* 150) refers to a Chaldean record of a lunar eclipse of 721 B.C. and to the division of the circle into 360°. The recognition of a zodiac of twelve signs, and the study of the courses of the planets, about 600 B.C., are further evidences of the interest of the Chaldean astronomers in this phase of applied mathematics.

Discussion 3

1. Influences favorable to the development of mathematics among the Greeks from 1000 B.C. to 300 B.C.

2. The nature of logistic and of arithmetic and the reasons for their treatment as unrelated subjects.

3. The advantages of the Greek method of treating arithmetic from the geometric standpoint, particularly in relation to the nature of irrational numbers.

4. The influence of Thales upon the subsequent development of mathematics in Greece.

5. The influences which contributed to the making of the character of Pythagoras.

6. The influence of Pythagoras upon mathematics in general, and particularly upon geometry and the theory of irrationals.

7. Music as a branch of ancient mathematics.

8. Beginnings of a kind of infinitesimal calculus in Greece, particularly with respect to the method of exhaustion.

9. Types of geometric propositions that attracted special attention in this period, thus showing the nature of geometry before the time of Euclid.

10. The influence of Plato upon mathematics in general and upon geometry in particular.

11. The influence of astronomy upon mathematics in Greece, particularly with reference to geometry and a primitive trigonometry.

12. The early steps in the invention of conic sections.

13. The study of higher plane curves among the Greeks in the period under discussion.

14. The influence of Aristotle upon mathematics in general, and particularly upon its applications.

15. Nature of mathematics in the Orient in this period.

16. General distinction between the mathematics of Greece and that of the East.

17. Mysticism of numbers as found in the Orient, in Mesopotamia, and in the West.

18. Early studies in the history of mathematics among the Greeks.

19. The recognition of the sphericity of the earth by various leading Greek philosophers.

20. The nature of the mathematics of the *Śulvasūtras*.