FOE1003
Introduction to Ordinary Differential Equations

Tadchanon Chuman
Department of Electrical Technology, SSRU

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Meaning and Types of ODEs

Meaning of ODEs

Ordinary differential equations (ODEs) consists of derivatives of some functions, e. g.,

$$\frac{dx}{dy} = f(x) \qquad \text{or} \qquad y' = f(x)$$

- y is a function of x,
- x is an independent variable,
- y is a dependent variable,
- dy/dx (or y′) denote the derivative of y with respect to x

Types of ODEs: Linearity

Note
y' : 1st-order derivative of y
y'' : 2nd-order derivative of y

ODE

Linear ODE

ODE such that y, y', y'' have no exponent greater than one and there are no cross-terms

Nonlinear ODE

ODE that is not linear

Two Notations of Derivatives

$\frac{dy}{dx} = y'$	1^{st}-order derivative
$\frac{d^2y}{dx^2} = y''$	2^{nd}-order derivative
$\frac{d^3y}{dx^3} = y''' = y^{(3)}$	3^{rd}-order derivative
$\frac{d^4y}{dx^4} = y^{(4)}$	4^{th}-order derivative

Example: Linearity

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(x)$$	This is a linear ODE because the terms d^2y/dx^2 dy/dx และ y are all linear.
$$y'' + 4yy' + 2y = \sin(x)$$	This is a nonlinear ODE because.....
$$y'' + cos(y) = 0$$	This is a...
$$(y'')^2 + 2y = \cos(x)$$	This is a...

Types of ODEs: Homogeneity

ODE

Homogeneous ODE

ODE of the form f(x,y)dy = g(x,y)dx such that the degree of f(x,y) and g(x, y) is same

Nonhomogeneous ODE

ODE that are not

Example: Homogeneity

$y'' + 4xy' + 2y = 0$	This is a homogeneous ODE because that is no forcing function.
$y'' + 4xy' + 2y = \sin(x)$	This is a nonhomogeneous ODE because …
$y'' \cos(x) + 4y' + 2xy = 0$	This is…
$y'' + \cos(y) = \mathrm{e}^x$	This is…

Exercises: Linearity and Homogeneity

ODE	Linearity	Homogeneity
1. $(y'')^2 - 3y' + 2y = x^4$		
2. $y'' + [a + b\sin(2x)]y = 0$		
3. $y''' - 6(y'')^2 + 11y' + 6y = \mathrm{e}^x$		
4. $y^4 = xy'' + y^2 = 0$		
5. $d(xy')/dx - xy = 0$		
6. $(x - y)dy = (x + y)dx$		

Order and Degree

ODEs may consists of several order of derivatives (e.g., y', y'', y''') where such derivatives may have multiple degrees(e.g., $(y')^3$, $(y'')^4$, $(y''')^2$)

Order of ODE is the highest order derivative it contains.

Degree) the degree of highest order derivative of the ODE.

Example: Order and Degree

ODE	Order	Degree
$y'' + 4xy' + 2y = 0$		
$(y''')^2 + 2x^2yy''' + x^2y = 0$		
$(y''')^2 + (y'')^3 + x^5y^8 = 0$		

Example: Order and Degree

ODE	Order	Degree
1. $(y'')^2 - 3y' + 2y = x^4$		
2. $y'' + [a + b\sin(2x)]y = 0$		
3. $y''' - 6(y'')^2 + 11y' + 6y = \mathrm{e}^x$		
4. $y^4 = xy'' + y^2 = 0$		
5. $d(xy')/dx - xy = 0$		
6. $(x - y)dy = (x + y)dx$		

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Solution of ODEs

Solution of ODEs

When we solve an ODE, the obtained result is called a **solution.**

Definition (Solution)

Solution is an expression for the dependent variable in terms of the independent one(s) which satisfies the relation

Example solutuin of the ODE : $y' = \cos(x)$ is $y = \sin(x)$ because

1. $y = \sin(x)$ has no derivative,
2. $y = \sin(x)$ corresponds to

$$y' = d \sin(x)/dx = \cos(x)$$

Types of ODE solutions

Solution of ODE

General Solution
Solution with an arbitrary constant, e.g., $y = Ae^{-x}$

Particular Solution
Obtained from a general solution, e.g., $y = 4e^{-x}$

Complete Solution
Linear combination of general solutions, e.g., $y = Ae^{-x} + B\sin(x)$

Homogeneous Solution
cannot be obtained from a general solution

Example: Types of Solutions

ODE	Solution	Solution type
$y'' - y = 0$	$y = Ae^{x}, \quad y = Be^{-x}$	general solutions ∵ has arbitrary constants
$y'' - y = 0$	$y = 3e^{-x}$	particular solution ∵ ιobtained by replacing B with 3
$y'' - y = 0$	$y = Ae^{x} + Be^{-x}$	complete solution ∵ combination of general solution
$(y')^2 - xy' + y = 0$	$y = cx - c^2, \quad y = x^2/4$	homogeneous solution ∵ $y=x^2/4$ is not obtained from general solutions

Solution Verification

When we think a function is a solution of an ODE, we can **verify** whether the function is the solution or not. The verification procedure is stated below.

1. Find derivatives of the function we want to verify.
2. Substitute the derivatives on one side of the ODE.
3. If both sides of the ODE are equal, we can conclude that the function is a solution of the ODE.

Example 1 Verify that $y = ae^{-x} + be^{2x}$ is a solution of

$$y'' - y' - 2y = 0$$

<u>Answer</u> Since there exist y′ and y″ in the equation, we have to derive the first- and second-derivatives of $y + be^{2x}$.

$$y' = \frac{d(ae^{-x} + be^{2x})}{dx} = ..$$

$$y'' = \frac{dy'}{dx} = \ldots$$

Example 1 Verify that $y = ae^{-x} + be^{2x}$ is a solution of

$$y'' - y' - 2y = 0$$

<u>Answer</u> (con.) Replacing y' and y'' in the LHS of the ODE yields

$$\mathrm{LHS} = y'' - y' - 2y = \ldots$$

Note
LHS : left-hand side
RHS : right-hand side

Example 2 Verify that $y = a\cos(2x) + b\sin(2x)$ is a solution of

$$y'' + 4y = 0$$

<u>Answer</u> Find the derivatives of y:

$$y''$$

$$y' =$$

$$y'' = dy'/dx =$$

$$\text{LHS} = y'' + 4y =$$

Example 3 Verify that $y^2 = ax - x\ln(x)$ is a solution of

$$2xyy' = (y^2 - x)$$

Answer Consider the LHS of the ODE.

$$\text{LHS} = 2xyy'$$

One can see that LHS = $x(2yy')$ and we know that $\dfrac{dy^2}{dx} = 2yy'$

Hence, $\text{LHS} = x\frac{dy^2}{dx} = \ldots$

Example 3 Verify that $y^2 = ax - x\ln(x)$ is a solution of

$$2xyy' = (y^2 - x)$$

Answer (con.) Consider the RHS of the ODE.

$$\text{RHS} = y^2 - x = \ldots$$

References

Adkins, William A., and Mark G. Davidson Ordinary differential equations. Springer Science & Business Media, 2012.

ศ. ดร.มงคล เดชนครินทร์ คณิตศาสตร์วิศวกรรมไฟฟ้า พิมพ์ครั้งที่ 4 สำนักพิมพ์จุฬาลงกรณ์มหาวิทยาลัย 2558.